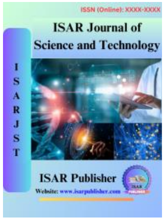

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On Matrix Graphs, Matrix Solutions of the Diophantine Equation $X_1^n + \cdots + X_m^n = X_{m+1}^n, n, m \geq 2$, Matrix Networks and von Neumann's Inequality for Complex Polynomials of Several Variables

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Abstract: We introduce the stability coefficients and stable sets of complex polynomials. We define matrix graphs and the construction structures set generated by a matrix graph. We introduce matrix networks linked to graph theory. We prove that any n-tuple of commuting contractions of matrix networks satisfies the von Neumann in equality. We define complex polynomials over N which don't have any positive integer roots but which have matrix roots with positive integers as entries. We show that these matrix roots are construction structures of matrix solutions of Diophantine equations. In particular, we show that the Diophantine equation $X_1^n + \cdots + X_m^n = X_{m+1}^n, n, m \geq 2$, admits an infinite number of matrix solutions with positive integers as entries.

Keywords: Matrices of integers, Diophantine equations.

Mathematics Subject Classification (2010): 15B36, 11D72.

1 Introduction and Main Result

Finding integer solutions of Diophantine equations has always been the main focus of Number Theory. Recent Mouanda's work on matrix solutions of Diophantine equations [1] leaves us without doubt that several Diophantine equations always admit an infinite number of matrix solutions. Finding matrix solutions with positive integers as entries of Diophantine equations becomes the main new research topic of the Galaxy Number Theory introduce by the author in 2021 [2].

In this paper, we introduce the stability coefficients and stable sets of complex polynomials. We introduce matrix graphs and the construction structures set. We also introduce matrix polynomials generated by matrix graphs called matrix networks. We prove that any n-tuple of commuting contractions of matrix networks satisfies the von Neumann inequality. We show that some particular complex polynomials over N which don't have any positive integer roots can have matrix roots with positive integers as entries. These matrix roots are construction structures of matrix solutions of Diophantine equations.

Theorem 1.1. *Let* $S_N = \{a_i : a_i \neq 0, a_i \neq 1;\ i = 1, \dots, n\}, n \geq 2$, *be a set of n prime numbers. Let* $(fs_n(x))_{n\geq 2}$ *be a sequence of complex polynomials over* N *defined by*

$$fs_n(x) = x^n - \prod_{i=1}^{n} a_i.$$

Then, the complex polynomial $fs_n(x)$ *does not have any positive integer roots for every n. However, the matrix Diophantine equation defined by*

$$fs_n(X) = 0, X \in M_n(\mathbb{N}),$$

admits at least $2n$ *matrix solutions for every n. Moreover, the Diophantine equation* $X_1^n + \cdots + X_m^n = X_{m+1}^n, \mathrm{n}, \mathrm{m} \geq 2$, *admits an infinite number of matrix solutions with positive integers as entries.*

2 Preliminaries

2.1 Rare Matrices

Definition 2.1. *A matrix* $B \in M_n(\mathrm{N})$ *is a construction structure of matrix solutions of Diophantine equations if there exist two positive integers* m, β *such that* $B^m - \beta \times I_n = 0$.

Denote by

$$D_n(\mathbb{N}) = \{B \in M_n(\mathbb{N}) : B^m - \beta \times I_n = 0, m, \beta \in \mathbb{N}\}$$

the set of all matrices of $M_n(\mathrm{N})$ which are construction structures of matrix solutions of Diophantine equations. A matrix Diophantine equation can admit several construction structures [3].

Definition 2.2. *[1]. The $n \times n$-matrices of the form*

$$c\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}, c\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}$$

$, a \neq 0, c \neq 0, b \neq 0, a, b, c \in \mathbb{C}$, are called Rare matrices of order n and index 1.

The index defines the number of non-zero complex coefficients of the matrix different to 1. It well known that Rare matrices have interesting properties.

Remark 2.3. *[1]. Let*

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C}), \alpha \neq 0,$$

be a Rare matrix of order n and index 1. Then

$$A_\alpha^n = \begin{pmatrix} \alpha & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \alpha \end{pmatrix}, A_\alpha^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \frac{1}{\alpha} \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_\alpha^{-1} = A_{\frac{1}{\alpha}}^T, A_\alpha^n = \alpha I_n, (\beta A_\alpha)^{-1} = \frac{1}{\beta} A_\alpha^{-1}, \beta \neq 0.$$

2.1 Graph Theory

In mathematics, the study of graphs is called graph theory. Graphs are mathematical structures used to model pairwise relations between objects. A graph in this context is made up of vertices (also called nodes or points) which are connected by edges (also called arcs, links or lines). A distinction is made between undirected graphs, where edges link two vertices symmetrically, and directed graphs, where edges link two vertices asymmetrically. Graphs are one of the principal objects of study in discrete mathematics, computer sciences, linguistics, physics and chemistry, social sciences, biology, mathematics (geometry, topology, Algebraic graph theory).

$$x\bullet \overset{edge}{\leftrightarrow} \bullet y$$

Definition 2.4. *A graph is an ordered triple $G = (V, E, \varphi)$ comprising*

- *V, a set of vertices (also called nodes or points);*
- *E, a set of edges (also called links or lines);*
- *$\varphi : E \to \{\{x, y\} : x, y \in V, x \neq y\}$ an incidence function mapping every edge to an unordered pair of vertices (that is, an edge is associated with two distinct vertices).*

In the edge $\{x, y\}$, the vertices x and y are called the endpoints of the edge. The edge is said to join x and y and to be incident on x and on y. A vertex may exist in a graph and not belong to an edge.

3 Stable Sets and Roots of Complex Polynomials

3.1 Stability Coefficients and Stable Sets of Complex Polynomials

Let

$$f(z) = \sum_{k=0}^{n} a_k z^k$$

be a complex polynomial over C and let α be a complex number. The set denoted by

$$S(\alpha) = \{z \in C : f(z) = \alpha\}$$

is called a stable set of the complex polynomial $f(z)$. The complex number α is called the stability coefficient of f. Let $(f_n(x))_{n\geq 1}$ be a sequence of complex polynomials over N defined by

$$f_n(x) = x^n.$$

For every positive integer n, the complex polynomial $f_n(x)$ does not have any stable set which has more than two elements. This observation could be different when the sequence $(f_n(X))_{n\geq 1}$ is a sequence of complex polynomials over $M_n(N)$ defined by

$$f_n(X) = X^n.$$

In this case, Rare matrices play an important role.

3.2 Sequences of Complex Polynomials over N without Positive Integer Roots

There are many ways we could construct sequences of complex polynomials over N^n without positive integer roots. The matrix roots of these polynomials allow us to compute the matrix solutions of certain Diophantine equations. Let $(f_n(x, y, z))_{n\geq 1}$ be a sequence of complex polynomials of three variables over N^3 defined by

$$f_n(x, y, z) = x^n + y^n - z^n.$$

The matrix roots of these complex polynomials allow us to find matrix solutions of the Fermat Diophantine equation. Recently, Mouanda proved that the Diophantine equation $x^n + y^n = z^n$ has an infinite number of matrix solutions. We want to investigate new types of matrix solutions induced by Rare matrices. Functional Analysis and Operator Theory both play an important role in this study. Let $(f_n(x))_{n\geq 1}$ be a sequence of complex polynomials over

Z defined by

$$f_n(x) = x^{2n} + 2n + 1.$$

Let $(E_n)_{n\geq 1}$ be a sequence of equations defined by

$$E_n : f_n(x) = 0.$$

In other words,

$$E_n = \{x \in Z : f_n(x) = 0\}.$$

The polynomial $f_n(x)$ does not have any integer root for every n.

This means that the set E_n does not have any element. We want to investigate the matrix roots of the polynomial $f_n(x)$. This require a deep understanding of the construction structures of matrices.

4 Construction of Matrix Graphs of order 7

In this section, we introduce the idea of matrix graphs which are mathematics objects called construction structures obtained by moving the entries of the first column, which are different to zero, between rows and drawing edges between the entries $a_{i,j}$ and $a_{k,j+1}, i \neq k$. Let α be a positive integer. Let

$$A_{\alpha,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order 7 and index 1. We can generate elements of the set D_7 (N) by moving α between rows of the matrix $A_{\alpha,1}$. Indeed,

$$A_{\alpha,2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,6} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,7} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can check that $A_{aj}^7 = a \times I_7, j = 1, 2, 3, 4, 5, 6, 7$. Therefore, the matrices $A_{\alpha,j}$ are elements of the set D_7(N). Every matrix $A_{\alpha,j}$ is called a matrix graph generated by the matrix $A_{\alpha,1}$. The set denoted by

$$Gra_7(A_{\alpha,1}) = \{A_{\alpha,j}, A_{\alpha,j}^T : j = 1, ..., 7, .., m\} \subset D_7(\mathbb{N})$$

is called the set of all matrix graphs generated by $A_{\alpha,1}$. As we said before, the elements of the set $Gra_7(A_{\alpha,1})$ are obtained by moving appropriately between rows the entries of the first column of the matrix $A_{\alpha,1}$ and drawing edges between the entries $a_{i,j}$ and $a_{k,j+1}$, $i \neq k$. Therefore, a matrix graph is a construction structure of matrix solutions of Diophantine equations.

4.1 Construction Structures Sets Generated by Matrix Graphs

In this section, we construct the elements of the set D_7(N). There are several ways of constructing the elements of the set D_7(N). The motion of the positive integer α between different columns inside the matrix $A_{\alpha,j}$ of the set $Gra_7(A_{\alpha,1})$, generates new elements of D_7(N) denoted by $A_{\alpha,j,k}$, $k = 1, 2, 3, ..., 7$. In other words, elements of the set $Gra_7(A_{\alpha,1})$ generate others elements of the set D_7(N). Indeed, consider the matrices $A_{\alpha,1,j}, j = 1, ..., 7$,

defined by

$$A_{\alpha,1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,1,2} = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,1,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,1,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,1,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,1,6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,1,7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is simple to verify that $A_{\alpha,1,k}^7 = \alpha \times I_7$. Therefore, the set

$$CS(A_{\alpha,1}) = \{A_{\alpha,1,k}, A_{\alpha,1,k}^T : k = 1, 2, 3, ..., 7\}$$

is called the construction structures set generated by the matrix $A_{\alpha,1}$. We could construct the set

$$CS(A_{\alpha,j}) = \{A_{\alpha,j,k}, A_{\alpha,j,k}^T : k = 1, 2, 3, ..., 7\}$$

which is the construction structures set generated by the matrix $A_{\alpha,j}$.

4.2 Construction Structures Induced by Finite Sets of Positive Integers

Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a finite set of positive integers. Suppose

$$A_{\alpha,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a Rare matrix of order 7 and index 1. The set S induced a new matrix $A_{\alpha,1}(S)$ defined by

$$A_{\alpha,1}(S) = \begin{pmatrix} 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in D_7(\mathbb{N}).$$

This matrix is obtained by just replacing the non-zero entry of the row i of the matrix $A_{\alpha,1}$ by the element a_i of the set S. The matrix $A_{\alpha,1}(S)$ is called the construction structure induced by the set S. The same thing could be done with the matrix

$$A_{\alpha,2}=\begin{pmatrix}0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\\alpha&0&0&0&0&0&0\\0&1&0&0&0&0&0\end{pmatrix}$$

In this case, we have

$$A_{\alpha,2}(S)=\begin{pmatrix}0&0&a_3&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_1&0&0&0&0&0&0\\0&a_2&0&0&0&0&0\end{pmatrix}.$$

We can construct the elements of the set $CS(A_{\alpha,1}(S))$ and $CS(A_{\alpha,2}(S))$ by moving a_1 between columns. Indeed, consider the matrices

$$A_{\alpha,1,1}(S)=\begin{pmatrix}0&a_2&0&0&0&0&0\\0&0&a_3&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_1&0&0&0&0&0&0\end{pmatrix}, A_{\alpha,1,2}(S)=\begin{pmatrix}0&a_1&0&0&0&0&0\\0&0&a_3&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_2&0&0&0&0&0&0\end{pmatrix}$$

$$A_{\alpha,1,3}(S)=\begin{pmatrix}0&a_3&0&0&0&0&0\\0&0&a_1&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_2&0&0&0&0&0&0\end{pmatrix}, A_{\alpha,1,4}(S)=\begin{pmatrix}0&a_3&0&0&0&0&0\\0&0&a_4&0&0&0&0\\0&0&0&a_1&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_2&0&0&0&0&0&0\end{pmatrix}$$

$$A_{\alpha,1,5}(S)=\begin{pmatrix}0&a_3&0&0&0&0&0\\0&0&a_4&0&0&0&0\\0&0&0&a_5&0&0&0\\0&0&0&0&a_1&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_2&0&0&0&0&0&0\end{pmatrix}, A_{\alpha,1,6}(S)=\begin{pmatrix}0&a_3&0&0&0&0&0\\0&0&a_4&0&0&0&0\\0&0&0&a_5&0&0&0\\0&0&0&0&a_6&0&0\\0&0&0&0&0&a_1&0\\0&0&0&0&0&0&a_7\\a_2&0&0&0&0&0&0\end{pmatrix},$$

$$A_{\alpha,1,7}(S)=\begin{pmatrix}0&a_3&0&0&0&0&0\\0&0&a_4&0&0&0&0\\0&0&0&a_5&0&0&0\\0&0&0&0&a_6&0&0\\0&0&0&0&0&a_7&0\\0&0&0&0&0&0&a_1\\a_2&0&0&0&0&0&0\end{pmatrix}.$$

Again, the construction of the matrix $A_{\alpha,1,k}(S)$ requires that the element a_1 of the S must occupy the column k. The set

$$CS(A_{\alpha,1}(S))=\left\{A_{\alpha,1,k}(S),A^T_{\alpha,1,k}(S):k=1,2,3,\dots,7\right\}$$

is called the construction structures set generated by the matrix $A_{\alpha,1}(S)$. Also, we have

$$P^7=\prod_{i=1}^{7}a_i\times I_7, \forall P\in CS(A_{\alpha,1}(S)).$$

We could also construct the set $CS(A_{\alpha,2}(S))$. Recall that

$$A_{\alpha,2}(S)=\begin{pmatrix}0&0&a_3&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\a_1&0&0&0&0&0&0\\0&a_2&0&0&0&0&0\end{pmatrix}, A_{\alpha,3}(S)=\begin{pmatrix}0&a_2&0&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\a_1&0&0&0&0&0&0\\0&0&0&0&0&0&a_7\\0&0&a_3&0&0&0&0\end{pmatrix}$$

We could construction structures sets

$$CS(A_{\alpha,2}(S))=\left\{A_{\alpha,2,k}(S),A^T_{\alpha,2,k}(S):k=1,2,3,\dots,7\right\}$$

and

$$CS(A_{\alpha,3}(S))=\left\{A_{\alpha,3,k}(S),A^T_{\alpha,3,k}(S):k=1,2,3,\dots,7\right\}.$$

In the same way, we could construct the set

$$CS(A_{\alpha,j}(S))=\left\{A_{\alpha,j,k}(S),A^T_{\alpha,j,k}(S):k=1,2,3,\dots,7\right\}$$

which is the construction structures set generated by the matrix $A_{\alpha,j}(S)$.

4.3 Applications

4.3.1 Matrix Solutions of the Diophantine Equation $X^7+Y^7=Z^7$

Recall that

$$A_{\alpha,6}=\begin{pmatrix}0&0&1&0&0&0&0\\\alpha&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\\0&1&0&0&0&0&0\end{pmatrix}, A_{\alpha,7}=\begin{pmatrix}0&0&1&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&1&0&0&0&0&0\\\alpha&0&0&0&0&0&0\end{pmatrix}$$

Let $S_1=\{a_1,a_2,a_3,a_4,a_5,a_6,a_7\}$ and $S_2=\{b_1,b_2,b_3,b_4,b_5,b_6,b_7\}$ be two finite sets of positive integers. We could define $A_{\alpha,6}(S_1)$ and $A_{\alpha,7}(S_2)$ as

$$A_{\alpha,6}(S_1)=\begin{pmatrix}0&0&a_3&0&0&0&0\\a_1&0&0&0&0&0&0\\0&0&0&a_4&0&0&0\\0&0&0&0&a_5&0&0\\0&0&0&0&0&a_6&0\\0&0&0&0&0&0&a_7\\0&a_2&0&0&0&0&0\end{pmatrix}, A_{\alpha,7}(S_2)=\begin{pmatrix}0&0&b_3&0&0&0&0\\0&0&0&0&0&0&b_7\\0&0&0&b_4&0&0&0\\0&0&0&0&b_5&0&0\\0&0&0&0&0&b_6&0\\0&b_2&0&0&0&0&0\\b_1&0&0&0&0&0&0\end{pmatrix}$$

It is possible to construct a graph defined by lines between the vertices $\{a_i,a_{i+1}\}$ and $\{b_i,b_{i+1}\}$.

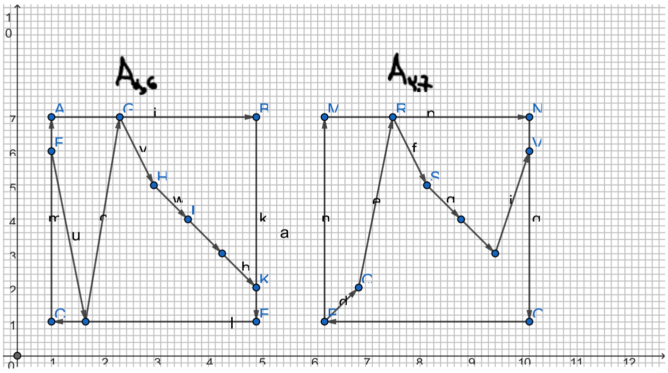

Figure 1: Graphs of Matrix Graphs

The graph of a matrix graph is the representation of the set of matrices with the same path. Suppose that

$$A_{\beta,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \beta = \prod_{i=1}^{7} a_i + \prod_{i=1}^{7} b_i.$$

We can deduce that

$$\begin{pmatrix} 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^7 + \begin{pmatrix} 0 & 0 & b_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_7 \\ 0 & 0 & 0 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6 & 0 \\ 0 & b_2 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^7 = A_{\beta,1}^7.$$

Therefore, the matrix triple $(A_{\alpha,6}(S_1), A_{\alpha,7}(S_2), A_{\beta,1})$ is a matrix solution of the Diophantine equation $X^7 + Y^7 = Z^7$.

Notation: Denote by

$$A_{\alpha,6}(\mathrm{N}) = \{A_{\alpha,6}(S_1) : S_1 \subset \mathrm{N}\},$$

$$A_{\alpha,7}(\mathrm{N}) = \{A_{\alpha,7}(S_2) : S_2 \subset \mathrm{N}\}$$

and

$$A_{\alpha,6}(\{1\}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,7}(\{1\}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let notice that $A_{\alpha,6}(\mathrm{N})$ is the set of matrix with the same graph.

Remark 4.1. *Let n be a positive integer and let* $S = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ *be a finite set of positive integers. Then,*

$$n\begin{pmatrix} 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^7 = \begin{pmatrix} 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ na_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^7.$$

This observation allows us to construct matrix solutions of the Diophantine equation

$$X_1^7 + X_2^7 + ... + X_n^7 = X_{n+1}^7, n \geq 2.$$

Indeed, assume that n = 10 and

$$A(S) = \begin{pmatrix} 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, B(S) = \begin{pmatrix} 0 & 0 & a_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 10a_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A simple calculation shows that the 10-tuple

$(A(S), A(S), A(S), A(S), A(S), A(S), A(S), A(S), A(S), B(S))$

is a matrix solution of the Diophantine equation

$$X_1^7 + X_2^7 + ... + X_{10}^7 = X_{11}^7.$$

4.3.2 Construction Structures and Integer Factorization

Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be a finite set of prime numbers such that $a_i < a_{i+1}$. Find the matrix the matrix $A(S) \in M_7(\mathrm{N})$ such that

$$A(S)^7 = \prod_{i=1}^{7} a_i \times I_7 = 3,212,440,751 \times I_7.$$

This matrix is an element of the set $D_7(\mathrm{N})$. A simple calculation shows that 13 × 17 × 19 × 23 × 29 × 31 × 37 = 3, 212, 440, 751.

Finally,

$a_1 = 13, a_2 = 17, a_3 = 19, a_4 = 23, a_5 = 29, a_6 = 31, a_7 = 37.$

$$A_{\alpha,5}(S) = \begin{pmatrix} 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_7 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 17 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 23 & 0 & 0 & 0 \\ 13 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 29 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 31 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 37 \\ 0 & 0 & 19 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.4 Matrix Networks (Matrix Polynomials) and von Neumann's Inequality for Complex Polynomials of Several Variables

In this section, we construct *n*-tuples of commuting contractions which satisfy the von Neumann in equality. Assume that $b_i = 1$ and let

$$X = A_{\alpha,7}(S_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a matrix graph. A simple calculation shows that

$$A = \sum_{k=0}^{6} c_k X^k = \sum_{k=0}^{6} c_k \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} c_0 & c_5 & c_1 & c_2 & c_3 & c_4 & c_6 \\ c_2 & c_0 & c_3 & c_4 & c_5 & c_6 & c_1 \\ c_6 & c_4 & c_0 & c_1 & c_2 & c_3 & c_5 \\ c_5 & c_3 & c_6 & c_0 & c_1 & c_2 & c_4 \\ c_4 & c_2 & c_5 & c_6 & c_0 & c_1 & c_3 \\ c_3 & c_1 & c_4 & c_5 & c_5 & c_0 & c_2 \\ c_1 & c_6 & c_2 & c_3 & c_4 & c_5 & c_0 \end{pmatrix}$$

This matrix is called a matrix network. We can now introduce the matrix network graph of A.

Network.pdf

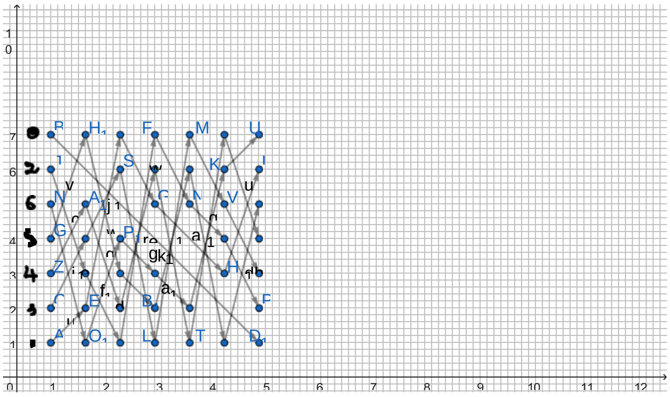

Figure 2: Matrix Network Graph

Let notice that inside the graph of a matrix network there is a graph of a matrix graph and matrix networks are matrix polynomials of one variable generated by matrix graphs. The set

$$\mathbb{V}_7(X) = \left\{ \begin{pmatrix} c_0 & c_5 & c_1 & c_2 & c_3 & c_4 & c_6 \\ c_2 & c_0 & c_3 & c_4 & c_5 & c_6 & c_1 \\ c_6 & c_4 & c_0 & c_1 & c_2 & c_3 & c_5 \\ c_5 & c_3 & c_6 & c_0 & c_1 & c_2 & c_4 \\ c_4 & c_2 & c_5 & c_6 & c_0 & c_1 & c_3 \\ c_3 & c_1 & c_4 & c_5 & c_6 & c_0 & c_2 \\ c_1 & c_6 & c_2 & c_3 & c_4 & c_5 & c_0 \end{pmatrix} : c_i \in \mathbb{C} \right\}$$

is called the set of all matrix networks generated by X. The set $\mathrm{V}_7(X)$ is commutative. Let

$$T_i = \begin{pmatrix} c_{0,i} & c_{5,i} & c_{1,i} & c_{2,i} & c_{3,i} & c_{4,i} & c_{6,i} \\ c_{2,i} & c_{0,i} & c_{3,i} & c_{4,i} & c_{5,i} & c_{6,i} & c_{1,i} \\ c_{6,i} & c_{4,i} & c_{0,i} & c_{1,i} & c_{2,i} & c_{3,i} & c_{5,i} \\ c_{5,i} & c_{3,i} & c_{6,i} & c_{0,i} & c_{1,i} & c_{2,i} & c_{4,i} \\ c_{4,i} & c_{2,i} & c_{5,i} & c_{6,i} & c_{0,i} & c_{1,i} & c_{3,i} \\ c_{3,i} & c_{1,i} & c_{4,i} & c_{5,i} & c_{5,i} & c_{0,i} & c_{2,i} \\ c_{1,i} & c_{6,i} & c_{2,i} & c_{3,i} & c_{4,i} & c_{5,i} & c_{0,i} \end{pmatrix}, i = 1, ..., n,$$

be n contractions of the set $\mathrm{V}_7(X)$. Let

$$f(z_1, ..., z_n) = \sum_{k_1=0,...,k_n=0}^{n_1,...,n_n} a_{(k_1,...,k_n)} z_1^{k_1} ... z_n^{k_n}$$

be a complex polynomial over D. Then, the n-tuple $\{T_1, ..., T_n\}$ of contractions satisfies the von Neumann inequality since all the elements of the set $\mathrm{V}_7(X)$ are matrix polynomials of one variable. That is,

$$\|f(T_1, ..., T_n)\| \leq \|f\|_\infty .$$

5 Construction of Matrix Graphs of order 8

Matrix graphs play an important role in the construction of matrix solutions of Diophantine equations. Let α be a positive integer. Let

$$A_{\alpha,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order 8 and index 1. We can generate elements of the set $D_8(\mathrm{N})$ by moving α between rows of the matrix $A_{\alpha,1}$. Indeed,

$$A_{\alpha,2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, A_{\alpha,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,7} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,8} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,9} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,10} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,11} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can check that $A_{aj}^8 = a \times I_8, j = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$.

The set

$$Gra_8(A_{\alpha,1}) = \left\{A_{\alpha,j}, A_{\alpha,j}^T : j = 1, ..., 10, 11, m\right\} \subset D_8(\mathbb{N})$$

is called the set of all matrix graphs generated by the matrix $A_{\alpha,1}$. We can construct structures sets $CS(A_{\alpha,j}) = \left\{A_{\alpha,j,k}, A_{\alpha,j,k}^T : k = 1, 2, 3, ..., 10, 11, ..., m\right\}$.

The elements of the set $CS(A_{\alpha,j})$ allow us to generate matrix solutions of the Diophantine equation $X_1^8 + ... + X_p^8 = X_{p+1}^8, p \geq 2$.

6 Construction of Matrix Graphs of order 9

In this section, we are not going to construct all the matrix graphs of order 9 but only the first 11 one. Let α be a positive integer. Let

$$A_{\alpha,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order 9 and index 1 [3]. We can generate elements of the set $D_9(\mathrm{N})$ by moving α between rows of the matrix $A_{\alpha,1}$. Indeed,

$$A_{\alpha,2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, A_{\alpha,3} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\alpha,6} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,8} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,9} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{\alpha,10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_{\alpha,11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We can notice that $A_{aj}^9 = a \times I_9$ We can deduce that $A_{\alpha,j} \in Gra_9(A_{\alpha,1})$. In fact,

$$Gra_9(A_{\alpha,1}) = \{A_{\alpha,j} : j = 1, 2, 3, \dots, 11, \dots, m\} \subset D_9(\mathrm{N}), m \geq 11.$$

We can construct structures sets $CS(A_{aj}) = \{A_{a,j,k}, A^T_{a,j,k} : k = 1,2,3, \dots ,10,11, \dots , m\}$. The elements of the set $CS\ (A_{\alpha,j})$ allow us to generate matrix solutions of the Diophantine equation $X_1^9 + \cdots + X_p^9 = X_{p+1}^9, p \geq\ 2$.

7 Main Result

In this section, we show that stable sets of matrix polynomials generate matrix solutions of Diophantine equations.

Proof of Theorem 1.1

Let $S_n = \{a_i : a_i \neq 0, a_i \neq 1; i = 1, \dots, n\}, n \geq 2$, be a set of n prime numbers. It is obvious that the complex polynomial $fs_n(x)$ over N defined by

$$f_{S_n}(x) = x^n - \prod_{i=1}^{n} a_i$$

does not have any positive integer roots for every n. Let $fs_n(x)$ be a matrix polynomial over $M_n(\mathrm{N})$ defined by

$$f_{S_n}(X) = X^n - \prod_{i=1}^{n} a_i \times I_n.$$

We need to show that the matrix polynomial $fs_n(x)$ admits a stable set. Let remind ourselves that matrices have different properties than positive integers. Let $\alpha \in$ N be a positive integer. Assume that

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

be a Rare matrix of order n_i and index 1. Then,

$$A_\alpha(S_n) = \begin{pmatrix} 0 & a_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{n-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & a_n \\ a_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The properties of Rare matrices allow us to observe that

$$A_\alpha(S_n)^n = \prod_{i=1}^{n} a_i \times I_n. \tag{7.1}$$

Define the matrix $A_\alpha(S_n, k)$ by exchanging a_1 and the entry a_k between columns. For example,

$$A_\alpha(S_n, 2) = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{n-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & a_n \\ a_2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $A_\alpha(S_n) = A_\alpha(S_n, 1)$ generates a set of n matrices $\{A_\alpha(S_n, k) : k = 1, \dots, n\}$ which satisfy the equation

$$A_\alpha(S_n, k)^n = \prod_{i=1}^{n} a_i \times I_n, k = 1, \dots, n. \tag{7.2}$$

Define the set $CS(A_{Sn}) = \{A_\alpha(S_n, k), A_\alpha(S_n, k)^T : k = 1, 2, \dots, n\} \subset D_n(\mathrm{N})$.

This set has a least $2n$ matrices. Therefore,

$$f_{S_n}(X) = X^n - \prod_{i=1}^{n} a_i \times I_n = 0, \forall X \in CS(A_{S_n}). \tag{7.3}$$

Finally, $CS(A_{Sn})$ is a stable set of the matrix polynomial $f_{Sn}(x)$ for every n. We can deduce that the $m + 1$ - tuple

$$(X, \dots, X, Y\ (S_n)), X\ \in CS(A_{Sn}\ )$$

with

$$Y(S_n) = \begin{pmatrix} 0 & a_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{n-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & a_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & a_n \\ ma_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are matrix solutions of the Diophantine equation

$$X_1^n + ... + X_m^n = X_{m+1}^n, n, m \geq 2.$$

References

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