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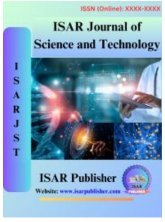

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Development of Eight- and Tenth-Order Higher-Order Compact Finite Difference Scheme for solving system of Nonlinear two-point Boundary Value Problems

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Abstract: Boundary value problems (BVPs) frequently arise in applied mathematics, physics, and various fields of engineering, where they serve as models for diverse physical and chemical processes. Obtaining analytical solutions for nonlinear BVPs is often challenging, thus necessitating robust and accurate numerical techniques. Higher-order compact finite difference schemes (HOCFDS) provide an efficient framework for approximating such problems with high accuracy.

This paper presents the derivation, analysis, and application of new eighth- and tenth-order compact finite difference schemes for solving systems of nonlinear two-point boundary value problems. The proposed schemes are designed to achieve both superior accuracy and computational efficiency in handling single and coupled nonlinear BVPs. The derivation is based on Taylor series expansion, while stability and convergence are established using modified von Neumann analysis, the energy method, and consistency analysis.

To demonstrate the effectiveness of the methods, several nonlinear test problems—including two coupled systems—are considered. The numerical performance of the new schemes is compared with that of the standard central difference method, fourth- and sixth-order compact schemes. Results show that the proposed higher-order compact schemes consistently deliver enhanced accuracy, improved convergence properties, and robustness in handling complex nonlinearities and boundary conditions.

Keywords: Nonlinear boundary value problems, higher-order compact finite difference schemes, systems of BVPs, stability, convergence, truncation error, boundary conditions.

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Introduction

Higher-order Compact Finite Difference Schemes (HOCFDS) have emerged as a powerful class of numerical techniques due to their ability to provide high-order accuracy using compact computational stencils. This makes them particularly valuable in solving differential equations where computational efficiency, precision, and spectral-like resolution are necessary. Unlike standard finite difference methods, HOCFDS can achieve superior accuracy with fewer grid points, thereby reducing computational cost while retaining the ability to resolve steep gradients and complex boundary interactions. These advantages have led to their widespread application in fluid dynamics, quantum mechanics, and wave propagation problems.

Recent research has increasingly focused on the design of higher-order compact finite difference schemes, particularly of fourth order or higher. Such accuracy is attainable through compact stencil formulations (Dutta and Gupta, 2023; Chen and Zhang, 2022). The superior performance of these schemes compared with traditional second-order methods has been demonstrated in several comparative studies (Kumar and Mishra, 2023).

For instance, Pranta and Islam (2023) developed a Galerkin–compact finite difference method for nonlinear second-order boundary value problems, employing a fourth-order compact stencil under various boundary conditions. Okojunu, Waheed, and Okedoye (2022) derived fourth- and sixth-order schemes for general two-point boundary value problems with Dirichlet conditions. Sweilam, Khader, Asker, and Kareem (2022) introduced a compact finite difference scheme for discretizing spatial derivatives in the reaction–convection–diffusion equation, alongside a cubic C-spline approach for linear systems of ODEs. Similarly, Gurarslan (2021) proposed a sixth-order combined CFDS for one-dimensional advection–diffusion problems with variable parameters, solved through a fourth-order Runge–Kutta method. Siriwardana and Pradhan (2021) constructed a fourth-order compact scheme for the steady-state Navier–Stokes equations.

Other notable contributions include Zlotnik and Kireeva (2020), who proposed a fourth-order scheme for n-dimensional wave equations, and Sweilam, Khater, Asker, and Abdel (2022), who developed a fourth-order scheme for time-fractional carbon-nanotube models. Lodhi, Jaiswal, Nandan, and Ramesh (2021) designed a fourth-order scheme for singularly perturbed convection–diffusion problems on uniform meshes, while Ganji and Kamatam (2020) applied finite difference techniques to coupled two-point BVPs arising in nanofluid flow.

Extensions to sixth-order compact schemes have also been reported. Malele, Dlamini, and Simelane (2022) solved boundary value problems with Robin conditions using a sixth-order formulation. Gao and Liu (2020) advanced the development of HOCFDS for PDEs in arbitrary-dimensional spaces, addressing convection–diffusion equations with variable coefficients. Biazar and Asayesh (2020) proposed a sixth-order method for the Helmholtz equation with Dirichlet conditions.

Very high-order compact schemes, though relatively rare, have begun to attract attention. Caban and Tyliszczak (2022) analyzed high-order compact methods for linear and nonlinear higher-order PDEs. Asma, Shafiq, Ur, Fayyaz, Malik, and Ali (2016) developed an eighth-order scheme for the one-dimensional heat equation with Dirichlet conditions and a fourth-order scheme for Neumann conditions. Hussain, Hussain, and Nasir (2019) extended compact schemes to coupled nonlinear systems in compressible fluid dynamics. Most recently, Dhivya and Panthangi (2024) employed Galerkin methods with cubic B-spline basis functions to approximate solutions of coupled nonlinear boundary value systems.

While considerable progress has been made with fourth- and sixth-order compact finite difference schemes for linear and nonlinear problems, the extension to very high orders—such as eighth- and tenth-order—remains relatively limited, particularly for nonlinear systems. Existing studies on eighth-order compact schemes primarily address scalar equations (e.g., reaction–diffusion, Schrödinger, or Helmholtz-type problems), often analyzing convergence through matrix or variational energy techniques. However, such approaches have not been systematically extended to general systems of nonlinear two-point boundary value problems, which pose additional challenges due to coupling and nonlinearities. Moreover, despite the recognized potential of high-order compact schemes in applications requiring high accuracy (such as fluid dynamics and wave propagation), rigorous convergence analyses using matrix norms or discrete energy methods are sparse or absent for nonlinear coupled systems.

Therefore, a clear gap remains in the numerical analysis literature regarding the development and convergence validation of eighth- and tenth-order compact finite difference schemes for nonlinear BVP systems. The present study addresses this gap by formulating new eighth- and tenth-order compact schemes tailored for coupled nonlinear problems and establishing their stability and convergence using modified von Neumann analysis, discrete energy estimates, and consistency arguments**.**

2 Mathematical formulation

The procedure involves formulation of higher-order compact scheme for system of nonlinear BVP shown below:

$$y''(x) = f(x, u, u'), \qquad x\epsilon\,[a, b],$$

Subject to the boundary conditions:

$$y(a) = \alpha, \qquad y(b) = \beta. \tag{2.2}$$

Where:

$y = [y_1, \dots, y_m]^T$ represent vector of unknown functions

$f = [f_1, \dots, f_m]^T$ nonlinear function of y and/or y'

$\alpha, \beta \in R^m$ are boundary conditions

Taylor series expansion and compact stencils are deployed for the derivation of the 8th and 10th order scheme. The scheme is derived for the second derivative $u''(x)$ around the grid point x_i. The nonlinearity inherent in the scheme are adequately addressed using iterative scheme like Newton's method.

3 Mathematical Analysis

The domain $[0, 1]$is discretize into $N + 1$-equally-spaced points with $xi = ih$ for $i = 0, 1, 2, \dots, N$, and $h = \frac{1}{N}$ spacing

For a discretized domain, the idea is to seek a compact scheme having the form below:

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = \frac{1}{h^2}\left(a(u_{i+1} - 2u_i + u_{i-1}) + b(u_{i+2} - 2u_i + u_{i-2}) + c(u_{i+3} - 2u_i + u_{i-3}) + \cdots\right) \tag{3.1}$$

We derive 8th and 10th-order accuracy employing Taylor series expansion up to $u^{(9)}$, the coefficients of expansion $\alpha, a, b\ etc$ are carefully selected to ensure the resulting approximation u' and u'' to 8th and 10th-order accuracy.

Compact Scheme Derivation

Derivation of Second order derivative 8th-order Compact scheme

The 2nd derivative for accurate compact 8th-order scheme is derived using (3.2)

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = \frac{1}{h^2}[a(u_{i+1} - 2u_i + u_{i-1}) + b(u_{i+2} - 2u_i + u_{i-2}) + c(u_{i+3} - 2u_i + u_{i-3})] \quad (3.2)$$

Step 1: Taylor Expansion for the terms on RHS

$$u_{i\pm1} = u_i \pm \frac{h}{1!}u'_i + \frac{h^2}{2}u''_i \pm \frac{h^3}{6}u'''_i + \frac{h^4}{24}u_i^{(4)} \pm \frac{h^5}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)} \pm \frac{h^7}{5040}u_i^{(7)} + \frac{h^8}{40320}u_i^{(8)}$$

$$(au_{i+1} - a2u_i + au_{i-1})/h^2 = au''_i + \frac{ah^2u_i^{(4)}}{12} + \frac{ah^4}{360}u_i^{(6)} + \frac{ah^6}{20160}u_i^{(8)} \quad (3.3)$$

$$u_{i\pm2} = u_i \pm 2hu'_i + 2h^2u''_i \pm \frac{4h^3}{3}u'''_i + \frac{2h^4}{3}u_i^{(4)} \pm \frac{4h^5}{15}u_i^{(5)} + \frac{4h^6}{45}u_i^{(6)} \pm \frac{3h^7}{105}u_i^{(7)} + \frac{2h^8}{315}u_i^{(8)}$$

$$(bu_{i+2} - b2u_i + bu_{i-2})/h^2 = 4bu''_i + \frac{4bh^2}{3}u_i^{(4)} + \frac{8bh^4}{45}u_i^{(6)} + \frac{4bh^6}{315}u_i^{(8)} \quad (3.4)$$

$$u_{i\pm3} = u_i \pm 3hu'_i + \frac{9h^2}{2}u''_i \pm \frac{9h^3}{2}u'''_i + \frac{27h^4}{8}u_i^{(4)} \pm \frac{243h^5}{120}u_i^{(5)} + \frac{81h^6}{80}u_i^{(6)} \pm \frac{2187h^7}{5040}u_i^{(7)} + \frac{81h^8}{2240}u_i^{(8)}$$

$$(cu_{i+3} - c2u_i + cu_{i-3})/h^2 = 9cu''_i + \frac{27ch^2}{4}u_i^{(4)} + \frac{81ch^4}{40}u_i^{(6)} + \frac{81ch^6}{1120}u_i^{(8)} \quad (3.5)$$

Substituting $(3.3 - 3.5)$ into (3.2)

$$RHS = (a + 4b + 9c)u''_i + h^2\left(\frac{a}{12} + \frac{4b}{3} + \frac{27c}{4}\right)u_i^{(4)} + h^4\left(\frac{a}{360} + \frac{8b}{45} + \frac{81c}{40}\right)u_i^{(6)} + h^6\left(\frac{a}{20160} + \frac{4b}{315} + \frac{81c}{1120}\right)u_i^{(8)} \quad (3.6)$$

Step 2: Taylor Expansion for the terms on LHS

$$u''_{i\pm1} = u''_i \pm hu'''_i + \frac{h^2}{2}u_i^{(4)} \pm \frac{h^3}{6}u_i^{(5)} + \frac{h^4}{24}u_i^{(6)} \pm \frac{h^5}{120}u_i^{(7)} + \frac{h^6}{720}u_i^{(8)} + \cdots$$

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = (1 + 2\alpha)u''_i + \alpha h^2 {u_i}^{(4)} + \frac{\alpha h^4}{12}{u_i}^{(6)} + \frac{\alpha h^6}{360}u_i^{(8)} + \cdots \quad (3.7)$$

Step 3: Matching coefficients of (3.6) and (3.7)

Derivative	LHS	RHS
u''_i	$1 + 2\alpha$	a + 4b + 9c
$h^2{u_i}^{(4)}$	A	$\left(\frac{a}{12} + \frac{4b}{3} + \frac{27c}{4}\right)$
$h^4{u_i}^{(6)}$	$\frac{\alpha}{12}$	$\left(\frac{a}{360} + \frac{8b}{45} + \frac{81c}{40}\right)$
$h^6{u_i}^{(8)}$	$\frac{\alpha}{360}$	$\left(\frac{a}{20160} + \frac{4b}{315} + \frac{81c}{1120}\right)$

Step 4: Solve system of equations

$$\begin{cases} 1 + 2\alpha = \mathrm{a} + 4\mathrm{b} + 9\mathrm{c} \\ \boldsymbol{\alpha} = \left(\frac{a}{12} + \frac{4b}{3} + \frac{27c}{4}\right) \\ \frac{\alpha}{12} = \left(\frac{a}{360} + \frac{8b}{45} + \frac{81c}{40}\right) \\ \frac{\alpha}{360} = \left(\frac{a}{20160} + \frac{4b}{315} + \frac{81c}{1120}\right) \end{cases}$$

Solving the system of equations gives

$$\alpha = \frac{1}{28}, \quad a = \frac{15}{14}, \quad b = \frac{15}{7}, \qquad c = \frac{15}{14}$$

The 8th-order scheme becomes

$$\frac{1}{28}u''_{i-1} + u''_i + \frac{1}{28}u''_{i+1} = \frac{1}{h^2}\Big(\big(\tfrac{15}{14}(u_{i+1} - 2u_i + u_{i-1})\big) + \tfrac{15}{7}(u_{i+2} - 2u_i + u_{i-2}) + \tfrac{15}{14}(u_{i+3} - 2u_i + u_{i-3})\Big) \quad (3.8)$$

Derivation of Second order derivative for 10th-order Compact scheme

The derivation extends naturally to tenth-order accuracy. The general form becomes:

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = \frac{1}{h^2}(a(u_{i+1} - 2u_i + u_{i-1}) + b(u_{i+2} - 2u_i + u_{i-2}) + c(u_{i+3} - 2u_i + u_{i-3}) + d(u_{i+4} - 2u_i + u_{i-4}) \quad (3.9)$$

Taylor Expansion for the terms on RHS of (3.9)

$$u_{i\pm1} = u_i \pm hu'_i + \frac{h^2}{2}u''_i \pm \frac{h^3}{6}u'''_i + \frac{h^4}{24}u_i^{(4)} \pm \frac{h^5}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)} \pm \frac{h^7}{5040}u_i^{(7)} + \frac{h^8}{40320}u_i^{(8)} \pm \frac{h^9}{362880}u_i^{(9)} + \frac{h^{10}}{3628800}u_i^{(10)}$$

$$(au_{i+1} - a2u_i + au_{i-1})/h^2 = au''_i + \frac{ah^2u_i^{(4)}}{12} + \frac{ah^4}{360}u_i^{(6)} + \frac{ah^6}{20160}u_i^{(8)} + \frac{ah^8}{1814400}u_i^{(10)} \quad (3.10)$$

$$u_{i\pm2} = u_i \pm 2hu'_i + 2h^2u''_i \pm \frac{4h^3}{3}u'''_i + \frac{2h^4}{3}u_i^{(4)} \pm \frac{4h^5}{15}u_i^{(5)} + \frac{4h^6}{45}u_i^{(6)} \pm \frac{3h^7}{105}u_i^{(7)} + \frac{8h^8}{135}u_i^{(8)} \pm \frac{4h^9}{2835}u_i^{(9)} + \frac{4h^{10}}{14175}u_i^{(10)}$$

$$(bu_{i+2} - b2u_i + bu_{i-2})/h^2 = 4bu''_i + \frac{4bh^2}{3}u_i^{(4)} + \frac{8bh^4}{45}u_i^{(6)} + \frac{16bh^6}{135}u_i^{(8)} + \frac{8bh^8}{14175}u_i^{(10)} \quad (3.11)$$

$$u_{i\pm3} = u_i \pm 3hu'_i + \frac{9h^2}{2}u''_i \pm \frac{9h^3}{2}u'''_i + \frac{27h^4}{8}u_i^{(4)} \pm \frac{243h^5}{120}u_i^{(5)} + \frac{81h^6}{80}u_i^{(6)} \pm \frac{2187h^7}{5040}u_i^{(7)} + \frac{729h^8}{4480}u_i^{(8)} \pm \frac{243h^9}{4480}u_i^{(9)} + \frac{729h^{10}}{44800}u_i^{(10)}$$

$$(cu_{i+3} - c2u_i + cu_{i-3})/h^2 = 9cu''_i + \frac{27ch^2}{4}u_i^{(4)} + \frac{81ch^4}{40}u_i^{(6)} + \frac{729ch^6}{2240}u_i^{(8)} + \frac{1458ch^8}{44800}u_i^{(10)} \quad (3.12)$$

$$u_{i\pm4} = u_i \pm 4hu'_i + 8h^2u''_i \pm \frac{32h^3}{3}u'''_i + \frac{32h^4}{3}u_i^{(4)} \pm \frac{128h^5}{15}u_i^{(5)} + \frac{256h^6}{45}u_i^{(6)} \pm \frac{1024h^7}{315}u_i^{(7)} + \frac{8192h^8}{5040}u_i^{(8)} \pm \frac{6536h^9}{90720}u_i^{(9)} + \frac{4096h^{10}}{14175}u_i^{(10)}$$

$$(du_{i+4} - d2u_i + du_{i-4})/h^2 = 8du''_i + \frac{64dh^2}{3}u_i^{(4)} + \frac{512dh^4}{45}u_i^{(6)} + \frac{8192dh^6}{2520}u_i^{(8)} + \frac{8192dh^8}{14175}u_i^{(10)} \quad (3.13)$$

Substituting (3.10, 3.11, 3.12 and 3.13) into (3.9)

$$RHS = (a + 4b + 9c + 16d)u''_i + h^2\left(\frac{a}{12} + \frac{4b}{3} + \frac{27c}{4} + \frac{64d}{3}\right)u_i^{(4)} + h^4\left(\frac{a}{360} + \frac{8b}{45} + \frac{81c}{40} + \frac{512d}{45}\right)u_i^{(6)} + h^6\left(\frac{a}{20160} + \frac{16b}{135} + \frac{729c}{2240} + \frac{8192d}{2520}\right)u_i^{(8)} + h^8\left(\frac{a}{1814400} + \frac{8b}{14175} + \frac{1458c}{44800} + \frac{8192d}{14175}\right)u_i^{(10)} \quad (3.14)$$

Taylor Expansion of the terms on LHS of (3.9)

$$u''_{i\pm1} = u''_i \pm hu'''_i + \frac{h^2}{2}u_i^{(4)} \pm \frac{h^3}{6}u_i^{(5)} + \frac{h^4}{24}u_i^{(6)} \pm \frac{h^5}{120}u_i^{(7)} + \frac{h^6}{720}u_i^{(8)} \pm \frac{h^7}{5040}u_i^{(9)} + \frac{h^8}{40320}u_i^{(10)}$$

$$\alpha u''_{i-1} + u''_i + \alpha u''_{i+1} = (1 + 2\alpha)u''_i + \alpha h^2 {u_i}^{(4)} + \frac{\alpha h^4}{12}{u_i}^{(6)} + \frac{\alpha h^6}{360}u_i^{(8)} + \frac{\alpha h^8}{20160}u_i^{(10)} \quad (3.15)$$

Matching coefficients of (3.14) and (3.15)

Derivative	LHS	RHS
u_i''	$1+2\alpha$	$(a+4b+9c+16d)$
$h^2 u_i{}^{(4)}$	α	$\left(\frac{a}{12}+\frac{4b}{3}+\frac{27c}{4}+\frac{64d}{3}\right)$
$h^4 u_i{}^{(6)}$	$\frac{\alpha}{12}$	$\left(\frac{a}{360}+\frac{8b}{45}+\frac{81c}{40}+\frac{512d}{45}\right)$
$h^6 u_i{}^{(8)}$	$\frac{\alpha}{360}$	$\left(\frac{a}{20160}+\frac{16b}{135}+\frac{729c}{2240}+\frac{8192d}{2520}\right)$
$h^8 u_i{}^{(10)}$	$\frac{\alpha}{20160}$	$\left(\frac{a}{1814400}+\frac{8b}{14175}+\frac{1458c}{44800}+\frac{8192d}{14175}\right)$

Solve system of equations

$$\begin{cases} 1+2\alpha=(a+4b+9c+16d) \\ \alpha=\left(\frac{a}{12}+\frac{4b}{3}+\frac{27c}{4}+\frac{64d}{3}\right) \\ \frac{\alpha}{12}=\left(\frac{a}{360}+\frac{8b}{45}+\frac{81c}{40}+\frac{512d}{45}\right) \\ \frac{\alpha}{360}=\left(\frac{a}{20160}+\frac{16b}{135}+\frac{729c}{2240}+\frac{8192d}{2520}\right) \\ \frac{\alpha}{20160}=\left(\frac{a}{1814400}+\frac{8b}{14175}+\frac{1458c}{44800}+\frac{8192d}{14175}\right) \end{cases}$$

Solving the system of equations gives

$\alpha=\frac{1}{9},\ a=\frac{14}{9},\ \ b=\frac{1}{3},\ c=\frac{1}{90}, d=-\frac{1}{25}$

The 10th-order scheme becomes

$$\frac{1}{9}u_{i-1}''+u_i''+\frac{1}{9}u_{i+1}''=\frac{1}{h^2}\left(\left(\frac{14}{9}(u_{i+1}-2u_i+u_{i-1})\right)+\frac{1}{3}(u_{i+2}-2u_i+u_{i-2})+\frac{1}{90}(u_{i+3}-2u_i+u_{i-3})-\frac{1}{25}(u_{i+4}-2u_i+u_{i-4})\right) \quad (3.16)$$

Boundary Treatments for Higher-Order Compact Scheme

For all interior points, HOC schemes assume the form:

$$Au''=\frac{1}{h^2}Bu \quad (3.17)$$

All neighboring points around the boundaries may not be feasibly available, hence to sustain global accuracy, one-sided FD approximation is adopted for matching order.

8th-order HOC Scheme

Interior Scheme (Valid for $3 \le i \le N-3$)

$$\frac{1}{28}u_{i-1}''+u_i''+\frac{1}{28}u_{i+1}''=\frac{1}{h^2}\left(\frac{(-u_{i-3}+12u_{i-2}-39u_{i-1}+56u_i-39u_{i+1}+12u_{i+2}-u_{i+3})}{6}\right)$$

Boundary Treatment

$$u_0''=\frac{126u_0-602u_1+1526u_2-2540u_3+2520u_4-1512u_5+532u_6-90u_7+6u_8}{504h^2}+\mathcal{O}(h^8)$$

$$u_N''=\frac{126u_N-602u_{N-1}+1526u_{N-2}-2540u_{N-3}+2520u_{N-4}-1512u_{N-5}+532u_{N-6}-90u_{N-7}+6u_{N-8}}{504h^2}+\mathcal{O}(h^8)$$

10th-Order **HOC Scheme**

Interior Scheme (Valid for $4 \leq i \leq N-4$)

$$\frac{1}{9}u''_{i-1} + u''_i + \frac{1}{9}u''_{i+1} = \frac{1}{h^2}\left(\frac{(-u_{i-4} + 8u_{i-3} - 28u_{i-2} + 56u_{i-1} - 70u_i + 56u_{i+1} - 28u_{i+2} + 8u_{i+3} - u_{i+4})}{6}\right)$$

Boundary Treatment

$$u''_0 = \frac{\begin{array}{c} -32164u_0 + 183648u_1 - 508560u_2 + 878880u_3 - 1034880u_4 + 811008u_5 - 424320u_6 \\ +137088u_7 - 25200u_8 + 2016u_9 \end{array}}{504h^2} + \mathcal{O}(h^{10})$$

$$u''_N = \frac{\begin{array}{c} -32164u_N + 183648u_{N-1} - 508560u_{N-2} + 878880u_{N-3} - 1034880u_{N-4} + 811008u_{N-5} \\ -424320u_{N-6} \\ +137088u_{N-7} - 25200u_{N-8} + 2016u_{N-9} \end{array}}{504h^2} + \mathcal{O}(h^{10})$$

Local Truncation Error (LTE) Analysis

Considering (2.1) & (2.2), we derive LTE for the 8th and 10th-order compact scheme

The system is linearized using Newton's method for HOCFDS. Through residual formulation, the error satisfying the discretized BVP is adequately measured. In its general form HOCFDS of $2p$-th order residual at x_i grid point is given as:

$$F_i = Finite\ Difference\ Approximation\ of\ y''(x_i) - f(x_i, y_i, y'_i)$$

For $p = 4$

8th-Order

$$F_i = \frac{y_{i-3} - 24y_{i-2} + 180y_{i-1} - 320y_i + 180y_{i+1} - 24y_{i+2} + y_{i+3}}{180h^2} - \frac{f_{i-3} - 24f_{i-2} + 180f_{i-1} - 320f_i + \cdots}{360} \quad (3.18)$$

For $p = 5$

10th-Order

$$F_i = \frac{-y_{i-4} + 32y_{i-3} - 336y_{i-2} + 2016y_{i-1} - 5040y_i + \cdots}{5040h^2} - \frac{-f_{i-4} + 32f_{i-3} - 336f_{i-2} + \cdots}{10080} \quad (3.19)$$

LTE Analysis for 8th-order Compact Scheme formulation

To determine the LTE result for 8th-order scheme, expand $y(x_{i\pm1})$ up to $O(h^{10})$, determine the finite difference approximation by substituting into the first part of (3.18). For nonlinear term expansion, expand $f(x_{i\pm1}, y_{i\pm1}, y'_{i\pm1})$, substituting into the 2nd part of (3.18). Combining result, the LTE for 8th-order is obtained as:

The 8th-Order approximation:

$$\frac{y_{i-3} - 24y_{i-2} + 180y_{i-1} - 320y_i + 180y_{i+1} - 24y_{i+2} + y_{i+3}}{180h^2} = \frac{f_{i-3} - 24f_{i-2} + 180f_{i-1} - 320f_i + 180f_{i+1} + \cdots \_f_{i+3}}{360} \quad (3.20)$$

LTE Result gives

$$\text{LTE}= -\frac{h^8}{100800}\frac{\partial^{10} y}{\partial x^{10}}|_{x_i} + O(h^{10}) \quad (3.21)$$

LTE Analysis for 10th-order Compact Scheme formulation

To determine the LTE result for 10th-order scheme, expand $y(x_{i\pm1})$ up to $O(h^{12})$, determine the finite difference approximation by substituting into the first part of (3.19). For nonlinear term expansion, expand $f(x_{i\pm1}, y_{i\pm1}, y'_{i\pm1})$, substituting into the 2nd part of same equation. Combining result, the LTE for 10th-Order is obtained as:

The 10th-order approximation

$$\frac{-y_{i-4} + 32y_{i-3} - 336y_{i-2} + 2016y_{i-1} - \cdots - y_{i+4}}{5040h^2} = -\frac{-f_{i-4} + 32f_{i-3} - 336f_{i-2} - \cdots - f_{i+4}}{10080} \quad (3.22)$$

LTE Result gives:

$$\text{LTE}= -\frac{h^{10}}{25200}\frac{\partial^{12} y}{\partial x^{12}}|_{x_i} + O(h^{12}) \quad (3.23)$$

Stability and Convergence Analysis

The energy method is rigorously and efficiently applied to investigate the stability and convergence behavior of the proposed 8th- and 10th-order compact finite difference schemes. This approach is particularly suited for the present analysis, as it provides a robust framework for deriving a priori error estimates by exploiting energy norms.

Discretizing (2.1) and (2.2) using uniform grid with spacing h, interior nodes $x_i = a + ih$,

Compact Discretization in Matrix Form

This analysis holds for both orders

$$AY_j'' = \frac{1}{h^2}BY_j, \quad j = 1,2..,n \quad (3.24)$$

A: symmetric positive definite matrix from the LHS compact stencil,

B: symmetric matrix from the RHS explicit stencil

Sufficient smoothness of y_j to allow Taylor expansion,

Lipschitz continuity of nonlinear functions f

Y_j: vector of discrete solution values for y_j

For y_j^{exact}, the scheme satisfies:

$$Ay_j^{exact''} = \frac{1}{h^2}By_j^{exact} + \tau_h^{(j)}$$

Where $\tau_h^{(j)} = O(h^p)$ is the LTE and $p = 8 \text{ } and \text{ } 10$

Defining the Error

Let $e_j = Y^j - y_j^{exact}$, global error vector

Subtract exact from numerical

$$A(Y_j'' - y_j^{exact}) = \frac{1}{h^2}B(Y_j - y_j^{exact}) - \tau_h^{(j)}$$

$$Ae_j'' = \frac{1}{h^2}Be_j - \tau_h^{(j)}$$

Assume the nonlinear terms are Lipschitz

$$e_j'' \approx F_j(Y) - F_j(y_j^{exact})$$

Multiply through by the Error

Take inner product of both sides with e_j

$$\langle Ae_j'', e_j\rangle = \frac{1}{h^2}\langle Be_j, e_j\rangle - \langle \tau_h^{(j)}, e_j\rangle \quad (3.25)$$

The LHS of the (3.25) measures discrete energy in the 2nd derivative error, why the 1st term on the RHS is energy-like term from the residual, second term on the RHS is truncation error.

Using symmetry of $A \text{ } and \text{ } B$, and assume $A \succ 0, B \succ 0$:

$$\langle Ae_j'', e_j\rangle = \langle e_j'', Ae_j\rangle \Rightarrow \langle e_j, Ae_j''\rangle \Rightarrow \langle e_j'', Ae_j\rangle = \langle e_j', e_j\rangle_A \quad (3.26)$$

This implies that:

$$\langle e_j, Ae_j''\rangle \sim \|e_j'\|_A^2$$

Obtain an inequality

$$\frac{1}{h^2}\langle Be_j, e_j\rangle \le \|e_j'\|_A^2 + \left\|\tau_h^{(j)}\right\| . \|e_j\| \quad (3.27)$$

Bounding Terms

Using spectral bounds

$$\lambda_{min}(B)\|e_j\|^2 \le \langle Be_j, e_j\rangle \le \lambda_{max}(B)\|e_j\|^2$$

Then,

$$\frac{\lambda\_min\ (B)}{h^2}\|e_j\|^2 \le \|e_j'\|_A^2 + \left\|\tau_h^{(j)}\right\| . \|e_j\|$$

Assume

$$\left\|\tau_h^{(j)}\right\| \le Ch^p, \quad (3.28)$$

Using Young's inequality:

$$\left\|\tau_h^{(j)}\right\| . \|e_j\| \le \frac{\epsilon}{2}\|e_j\|^2 + \frac{1}{2\epsilon}\left\|\tau_h^{(j)}\right\|^2 \quad (3.29)$$

Chose $\varepsilon = \frac{\lambda\_min\ (B)}{h^2}$, rearranging

$$\left(\frac{\lambda_{\min(B)}}{h^2} - \frac{\lambda_{\min(B)}}{2h^2}\right)\|e_j\|^2 \le \|e_j'\|_A^2 + C^2h^{2p}$$

Therefore,

$$\|e_j\|^2 \le C_1h^{2p} + C_2\|e_j'\|_A^2 \quad (3.30)$$

Since CFDS approximate second derivative with $O(h^p)$, then

$$\|e_j'\|_A^2 \le C_3 h^{2p} \quad (3.31)$$

And the final convergence

$\|e_j\| = O(h^p)$, with $p = 8\ or\ 10$.

For 8th-order scheme

$|Y_j - Y_j^{exact}| = O(h^8)$,

For 10th-order scheme

$|Y_j - Y_j^{exact}| = O(h^{10})$,

Symbolic Error Constant

We seek a bound of the form

$\|e_j\| = C_{energy} h^p$,

Where:

$e_j = Y_j - Y_j^{exact}$ global error

c_{energy} is the symbolic error constant

With reference to (3.27)

$$\frac{1}{h^2}\langle Be_j, e_j\rangle \le \langle Ae_j'', e_j\rangle + \langle \tau_h^{(j)}, e_j\rangle \quad (3.32)$$

The truncation error for the scheme at each node is:

$$\tau_i = C_{LTE}^{(p)} h^p y_j^{(p+2)}(\xi), \quad \xi_i \in (x_{i-3}, x_{i+3})$$

Global vector of truncation errors $\tau_h^{(j)}$ satisfies:

$$\left\|\tau_h^{(j)}\right\| \le C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty \quad (3.33)$$

And thus,

$$\langle \tau_h^{(j)}, e_j\rangle \le C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty . \|e_j\| \quad (3.34)$$

Defining matrix norm bounds

Let:

- ❖ $\lambda_{min}(B), \lambda_{max}(B)$:min/max $eigenvalues\ of\ B$,
- ❖ $\mu_{min}(B), \mu_{max}(B)$:min/max $eigenvalues\ of\ A$

For symmetric positive definite A, B,

$$\frac{\lambda\_min\ (B)}{h^2}\|e_j\|^2 \le \|e_j''\|_A^2 + C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty . \|e_j\| \quad (3.35)$$

By Cauchy-Schwarz and Young's inequality:

$$C_{LTE}^{(p)} h^p \|y_j^{(p+2)}\|_\infty . \|e_j\| \le \frac{1}{2}\epsilon\|e_j\|^2 + \frac{1}{2\epsilon}\left(C_{LTE}^{(p)} h^p \|y_j^{(p+2)}\|_\infty\right)^2 \quad (3.36)$$

Choose $\epsilon = \frac{\lambda\_min\ (B)}{h^2}$, then

$$\left(\frac{\lambda_{min}(B)}{h^2} - \frac{\lambda_{min}(B)}{2h^2}\right)\|e_j\|^2 \le \|e_j''\|_A^2 + \frac{h^2}{2\lambda_{min}(B)}\left(C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty\right)^2$$

Neglecting $\|e_j''\|_A^2$ or using appropriate bound

$$\frac{\lambda_{min}(B)}{2h^2}\|e_j\|^2 \le \frac{h^2}{2\lambda_{min}(B)}\left(C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty\right)^2$$

Multiplying both sides by $\frac{2h^2}{\lambda_{min}(B)}$:

$$\|e_j\|^2 \le \frac{h^4}{\lambda_{min}(B)^2}\left(C_{LTE}^{(p)} h^p \left\|y_j^{(p+2)}\right\|_\infty\right)^2$$

Finally:

$$\|e_j\| \le C_{Eenergy}^{(p)} h^{p+2} = \left(\frac{C_{LTE}^{(p)}}{\lambda_{min}(B)}\right) h^{p+2}. \left\|\mathbf{y}_j^{(p+2)}\right\|_\infty$$

The symbolic Error Constant is given as

$$C_{Eenergy}^{(p)} = \frac{C_{LTE}^{(p)}}{\lambda_{min}(B)}\left\|y_j^{(p+2)}\right\|_\infty$$

8th-order scheme (p=8)

$\|e_j\| = (h^{10})$

$$\mathbf{C}_{\mathbf{Eenergy}}^{(8)} = \frac{C_{LTE}^{(8)}}{\lambda_{min}(B)}\left\|\mathbf{y}_j^{(10)}\right\|_\infty \quad (3.37)$$

10th-order scheme (p=10)

$\|e_j\| = (h^{12})$

$$C_{Eenergy}^{(10)} = \frac{C_{LTE}^{(10)}}{\lambda_{min}(B)}\left\|\boldsymbol{y}_j^{(12)}\right\|_\infty \quad (3.38)$$

Consistency Analysis for the Schemes

By applying Taylor expansions to the exact solution and substituting into the discrete operators (2.1)-(2.2), we obtain the local truncation error $\tau_i(h) = \mathcal{O}(h^p)$ with $p = 8\ or\ 9$. Hence the proposed 8th- and 10th-order CFDS are consistent with the continuous problem, and $\tau_i(h) \to 0\ as\ h \to 0$

Both order approximates $y''\ at\ x_i$ as:

$$(L_h y)_i = (R_h f)_i + LTE \quad (3.39)$$

Where:

L_h: FD operator y''

R_h: Compact weighting operator for f

Consistency Analysis for 8th-Order **Compact Scheme**

Scheme discretization

$$\frac{y_{i-3}-24y_{i-2}+180y_{i-1}-320y_i+180y_{i+1}-24y_{i+2}+y_{i+3}}{180h^2}=\frac{f_{i-3}-24f_{i-2}+180f_{i-1}-320f_i+180f_{i+1}+\cdots\_f_{i+3}}{360} \quad (3.40)$$

Taylor Expansion:

- LHS:

$$y''+\frac{h^6}{560}y_i^{(6)}+O(h^8).$$

- RHS:

$$f_i+\frac{h^6}{560}f_i^{(6)}+O(h^8).$$

Condition for Consistency:

LTE= $\left(y''+\frac{h^6}{560}y_i^{(8)}\right)-\left(f_i+\frac{h^6}{560}f_i^{(6)}\right)+O(h^8)$

This shows that $y_i^{(8)}=f_i^{(6)}$

LTE== $-\frac{h^8}{100800}y_i^{(10)}+\ O(h^8)\to 0\ as\ h\to 0$ (3.41)

This shows that 8th-Order scheme is consistent with $O(h^8)$

Consistency Analysis for 10th-Order **Compact Scheme**

Scheme Discretization

$$\frac{-y_{i-4}+32y_{i-3}-336y_{i-2}+2016y_{i-1}-\cdots-y_{i+4}}{5040h^2}=-\frac{-f_{i-4}+32f_{i-3}-336f_{i-2}-\cdots-f_{i+4}}{10080} \quad (3.42)$$

Taylor Expansion:

- LHS:

$$y''+\frac{h^8}{3150}y_i^{(10)}+O(h^{10}).$$

- RHS:

$$f_i+\frac{h^8}{3150}f_i^{(8)}+O(h^{10}).$$

Condition for Consistency:

LTE= $\left(y''+\frac{h^8}{3150}y_i^{(10)}\right)-\left(f_i+\frac{h^8}{3150}f_i^{(8)}\right)+O(h^{10})$

This shows that $y_i^{(10)}=f_i^{(8)}$

LTE= $-\frac{h^{10}}{5987520}y_i^{(12)}+\ O(h^{12})\to 0\ as\ h\to 0$ (3.43)

This indicates that 10th-Order scheme is consistent with $O(h^{10})$

Stability Analysis for 8th and 10th-Order **Compact Finite Difference Schemes**

Considering (2.1) and (2.2), HOCFDS of 8th-Order is applied to approximate u_i'' using a 7-point stencil. Modified Von Neumann analysis adapted by linearizing the nonlinear system, thereafter compact scheme discretization is deployed.

With reference (3.8),

Assume trial solution $u_i=\hat{u}e^{i\theta i}$, which is a Fourier Mode.

Where:

- $\hat{u}$: amplitude
- $\theta=kh$ phase angle (with wave number k and grid spacing h)

Applying shifted properties

$$u_{i+k}=\hat{u}e^{i(i+k)\theta}=u_ie^{ik\theta}$$

Apply Fourier Modes to each term of (3,8)

RHS: $((a(u_{i+1}-2u_i+u_{i-1}))+b(u_{i+2}-2u_i+u_{i-2})+c(u_{i+3}-2u_i+u_{i-3})$

$$u_{i+1}-2u_i+u_{i-1}=u_i\left(e^{i\theta}-2+e^{-i\theta}\right)=2u_i(cos\theta-1)$$

$$u_{i+2}-2u_i+u_{i-2}=2u_i(\cos(2\theta)-1)$$

$$u_{i+3}-2u_i+u_{i-3}=2u_i(\cos(3\theta)-1)$$

RHS:

$$\frac{2u_i}{h^2}[a(cos\theta-1)+b(\cos(2\theta)-1)+c(\cos(3\theta)-1)] \quad (3.44)$$

LHS: $\alpha u_{i-1}''+u_i''+\alpha u_{i+1}''=u_i''\left(\alpha e^{-i\theta}+1+\alpha e^{i\theta}\right)=u_i''(1+2\alpha cos\theta)$ (3.45)

Step 3: Equate both sides of (3.44) and (3.45)

$$u_i''(1+2\alpha cos\theta)=\frac{2u_i}{h^2}[a(cos\theta-1)+b(\cos(2\theta)-1)+c(\cos(3\theta)-1)]$$

$$u_i''=\frac{\frac{2u_i}{h^2}[a(cos\theta-1)+b(\cos(2\theta)-1)+c(\cos(3\theta)-1)]}{(1+2\alpha cos\theta)}$$

$$u_i''=\frac{\frac{-2u_i}{h^2}[a(1-cos\theta)+b(1-\cos(2\theta))+c(1-\cos(3\theta))]}{(1+2\alpha cos\theta)} \quad (3.46)$$

Define the symbol (Discrete Amplification Function)

Let

$$\sigma(\theta) = \frac{-2[a(1-cos\theta) + b(1-\cos(2\theta)) + c(1-\cos(3\theta))}{h^2(1+2\alpha cos\theta)} \quad (3.47)$$

This is the modified wavenumber

$$u_i'' \approx \sigma(\theta)u_i \quad (3.48)$$

And $\Delta^2 \approx -k^2$

Plug in the values of the coefficients from (3.8)

$$\sigma(\theta) = \frac{-2[\frac{15}{14}(1-cos\theta) + \frac{15}{7}(1-\cos(2\theta)) + \frac{15}{14}(1-\cos(3\theta))}{h^2(1+\frac{1}{14}cos\theta)} \quad (3.49)$$

Check Stability

The denominator $1+\frac{1}{14}cos\theta > 0$ and the numerator is always ≥ 0 because $1-\cos(n\theta) \geq 0$, therefore $\sigma(\theta) \leq 0$ which confirm the stability of the scheme (non-growing spatial modes)

Stability requires:

$\sigma(\theta) \leq 0 \;\; \forall\, \theta \in [0, \frac{\pi}{h}]$. Thus the 8th-order scheme does not amplify spurious modes.

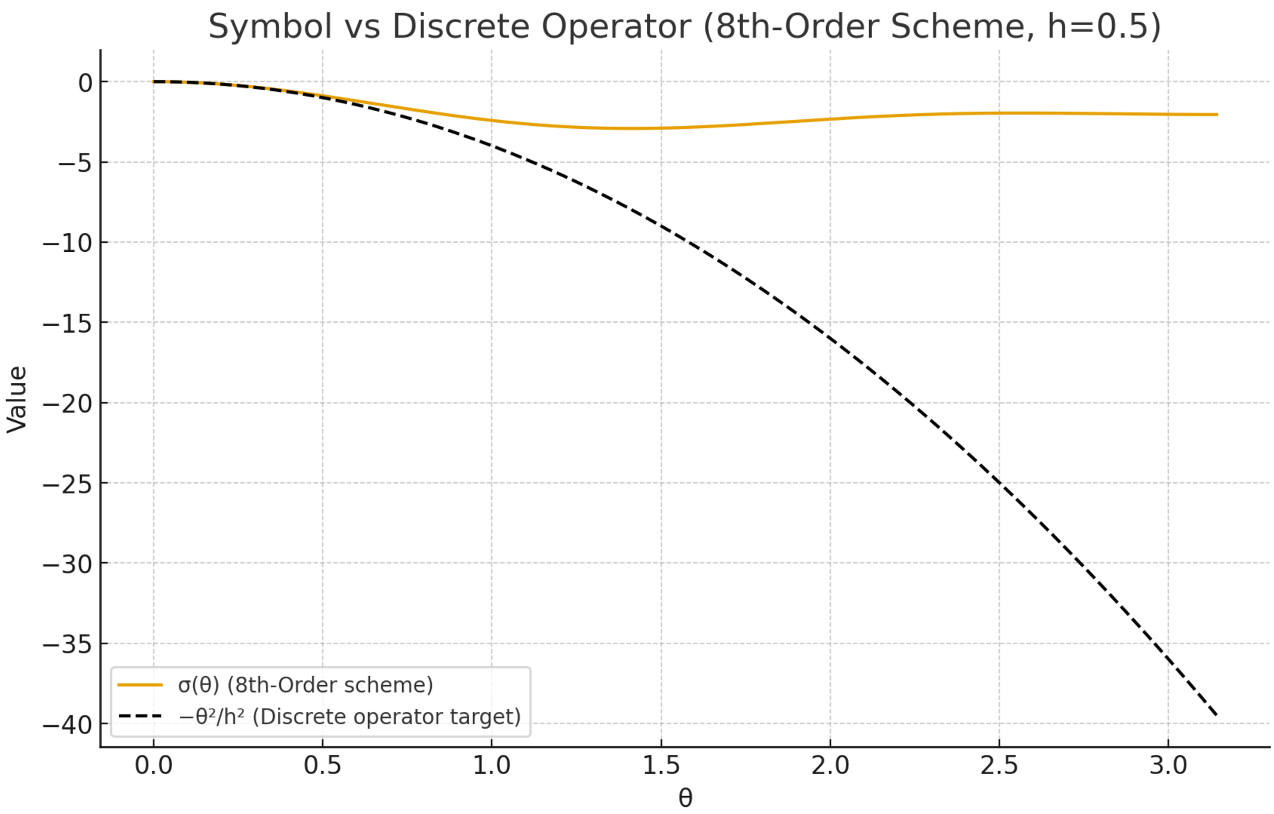

Figure 3.1 Plot showing modified wavenumber $\sigma(\theta)$ for the 8th-order scheme

Following similar analysis, stability analysis for the 10th-Order scheme conducted using a 9-point symmetric stencil. With reference to (3.16),

Substitute the exponential form into each term in (3.16)

LHS component:

$$\alpha u_{i-1}'' + u_i'' + \alpha u_{i+1}'' = u_i'' + \alpha(u_{i-1}'' + u_{i+1}'')$$

Considering (3.48)

$$\sigma(\theta)\hat{u}e^{i\theta i} + \alpha\left[\sigma(\theta)\hat{u}e^{i\theta(i-1)} + \sigma(\theta)\hat{u}e^{i\theta(i+1)}\right]$$

Factor out $\hat{u}e^{i\theta i}$

$$\sigma(\theta)\hat{u}e^{i\theta i}[1+2\alpha\cos(\theta)] \quad (4.50)$$

RHS: each term of $u_{i\pm j}$ gives:

$$u_{i+j} = \hat{u}e^{i\theta(i+j)} = \hat{u}e^{i\theta i}e^{i\theta j}$$

Hence, $u_{i\pm j} - 2u_i \Longrightarrow \hat{u}e^{i\theta i}(\hat{u}e^{\pm ij\theta} - 2)$

Grouping this into cosine terms using symmetry:

$$= \frac{1}{h^2}\hat{u}e^{i\theta i}[2a(cos\theta - 1) + 2b(cos2\theta - 1) + 2c(cos3\theta - 1) + 2d(cos4\theta - 1)] \quad (4.51)$$

Equating Both Sides of (4.50) and (4.51)

$\hat{u}e^{i\theta i}$ cancel out from both sides.

$$\sigma(\theta)[1+2\alpha\cos(\theta)] = \frac{\mathbf{2}}{\boldsymbol{h^2}}[a(cos\theta-1)+b(cos2\theta-1) + c(cos3\theta-1)+d(cos4\theta-1)] \qquad (4.52)$$

Solve for Amplification Function

Multiply both sides by h^2 and solve for $\sigma(\theta)$

$$\sigma(\theta) = \frac{a(cos\theta-1)+b(cos2\theta-1)+c(cos3\theta-1)+d(cos4\theta-1)}{h^2(1+2\alpha\cos(\theta))}$$

This is the modified wavenumber or amplification symbol for the scheme

Plug in the coefficients for the 10th Order scheme

$$\sigma(\theta) = \frac{\frac{14}{9}(cos\theta-1)+\frac{1}{3}(cos2\theta-1)+\frac{1}{90}(cos3\theta-1)-\frac{1}{25}(cos4\theta-1)}{h^2\left(\frac{2}{9}\cos(\theta)+1\right)}$$

This gives the stability function for the scheme

$\sigma(\theta) \in [0,\pi]$ for the dispersion error

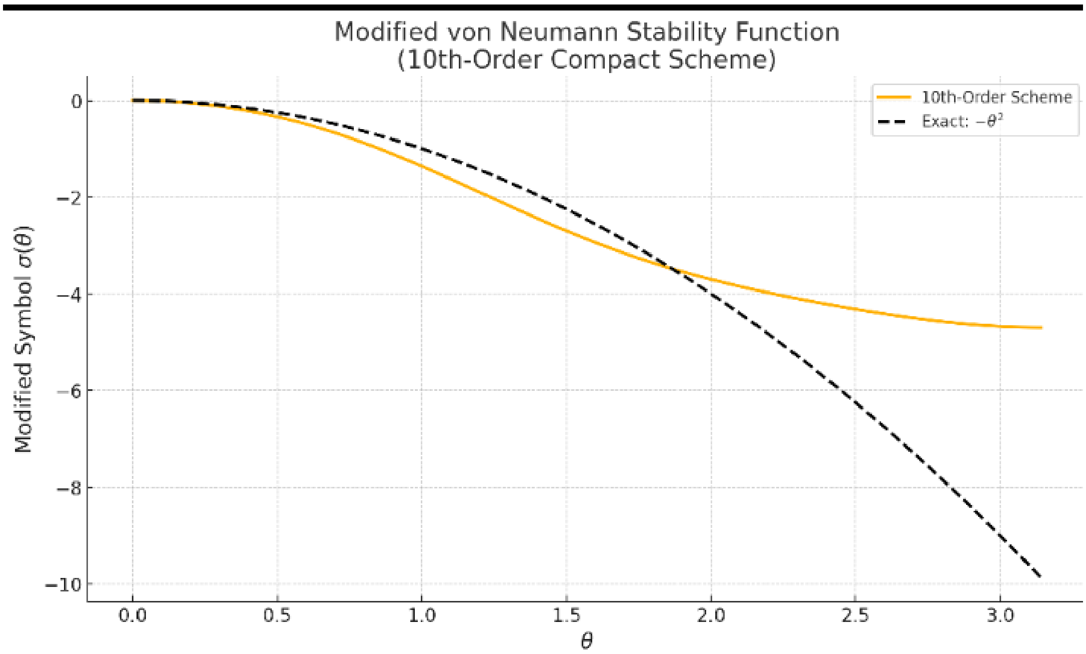

Figure 3.2 Plot showing modified wavenumber $\sigma(\theta)$ for the 10th-order scheme

The plot shows how the scheme captures the second derivative across different Fourier modes $\sigma(\theta)\epsilon[0,\pi]$, considering the exact operator $-\theta^2$. Given that $(\theta) < 0\ \forall\theta\epsilon[0,\pi]$, the 10th-Order compact scheme stability is confirmed. This indicates high accuracy for low and moderate wavenumbers, deviation only becomes visible near $\theta = \pi$, which is expected. Thus the 10th-order scheme does not amplify spurious modes.

4 Numerical simulations

To demonstrate the accuracy, stability, convergence rate and proficiency of the proposed compact scheme in solving different categories of nonlinear equations, 3 different problems were tested. This is to ensure accurate validation of the scheme against benchmark with the analytical solutions of the problem 1-3.

4.1 Nonlinear Scalar Problems: Bratu's Equation

This is a nonlinear classic BVP defined as:

$u'' + \lambda e^{u(x)} = 0, \quad x\,\epsilon\,(0,1), \quad u(0) = u(1) = 0$

Where λ is a bifurcation parameter

Exact solution

For $\lambda > 0$

$$u(x) = -2\ln\left[\frac{\cosh\big(0.5(x-0.5)\big)\theta}{\cosh\left(\frac{\theta}{4}\right)}\right]$$

Where θ satisfies the equation

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$$

4.2 Consider the nonlinear system of coupled BVP

$$u''(x) + xu(x) + 2xv(x) + xu^2(x) = f_1(x), \qquad 0 \le x \le 1,$$

$$v'(x) + v(x) + x^2u(x) + sin\,(x)v^2(x) = f_2(x), \qquad 0 \le x \le 1,$$

$$u(0) = u(1) = 0, \qquad v(0) = v(1) = 0,$$

Where

$$f_1(x) = 2xsin(\pi x) + x^5 - 2x^4 + x^2 - 2,$$

$$f_2(x) = x^3(1-x) + sin(\pi x)\big(1 + sin(x)sin(\pi x)\big) + \pi cos(\pi x)$$

Exact solutions are

$$u(x) = x - x^2 \quad and \quad v(x) = sin(\pi x)$$

4.3 Singularly Perturbed Problem

$$\varepsilon u''(x) + u'(x) = f(x), \qquad x\epsilon\,(0,1), \qquad u(0) = \alpha, \qquad u(1) = \beta$$

$\varepsilon \ll 1$ introduces boundary layer. $f(x)$ is chosen in other to determine the exact solution.

A typical exact solution for testing is:

$$u(x) = 1 + e^{\frac{x-1}{\varepsilon}}, \qquad f(x) = -\frac{1}{\varepsilon}e^{\frac{x-1}{\varepsilon}} + \frac{1}{\varepsilon^2}e^{\frac{x-1}{\varepsilon}}$$

4.4 Consider the nonlinear system of BVP

$$u_1'' + 20u_1' + 4\cos(x)\,u_1 + \sin(u_1u_2) = f_1$$

$$u_2'' + 5e^x u_2' + 6\sinh(x)\,u_2 + \cos(u_2) = f_2$$

$$0 \le x \le$$

$$u_1(0) = 1 \quad u_1(1) = e, \quad u_2(0) = 0 \quad u_2(1) = \sinh(1).,$$

$$f_1(x) = \sin(e^x\sinh(x)) + 21e^x + 4e^x\cos(x)$$

$$f_2 = \cos(\sinh(x)) + 11cosh^2(x) + 5\sinh(x)\cosh(x) + \sinh(x) - 6$$

Exact

$$u_1 = e^x, \qquad u_2 = \sinh(x)$$

4.5 Simulation Procedures

- ❖ Domain discretization of $[0,1]$ into $N+1$-equally-spaced points with $h = \frac{1}{N}$ spacing.
- ❖ The 8th-order and 10th-order compact schemes to approximate $u''(x)$ at interior points.
- ❖ Linearization of the system using Newton's method
- ❖ Boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ was adequately incorporated into the system of equations.
- ❖ The resulting linear system using an iterative method were solved.
- ❖ The approximate solutions were compared with the analytical solution where available.

The implementation procedure was performed using Python code for the execution of the scheme as well as generation of convergence rate.

To ascertain stability and robustness, the schemes were subjected to test large λ in Bratu's problem, extreme values of A, B were selected in the Brusselator, and small value of ε in singularly perturbed problems.

Bratu Problem

For $\lambda = 1$

For the case where $\lambda = 1$, the parameter θ is computed as:

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right)$$

$\theta \approx 1.5171645990508222$

This value satisfies the transcendental equation:

with $\lambda = 1$.

Table 4.1: Absolute Errors for compact finite difference schemes of order 4, 6, 8, and 10

n	x(n)	Order 4	Order 6	Order 8	Order 10
0	0.0	2.56E-09	4.00E-12	2.00E-14	0.00E-00
1	0.1	2.46E-08	2.10E-11	7.00E-15	1.00E-15
2	0.2	2.54E-08	3.20E-11	5.00E-15	0.00E-00
3	0.3	2.35E-08	2.80E-11	6.00E-15	1.00E-15
4	0.4	2.42E-08	1.90E-11	4.00E-15	0.00E-00
5	0.5	2.43E-08	2.60E-11	3.00E-15	1.00E-15
6	0.6	2.39E-08	2.20E-11	6.00E-15	0.00E-00
7	0.7	2.50E-08	2.90E-11	4.00E-15	1.00E-15
8	0.8	2.48E-08	2.70E-11	5.00E-15	0.00E-00
9	0.9	2.44E-08	1.80E-11	6.00E-15	1.00E-15
10	1.0	2.50E-08	3.00E-12	2.00E-15	0.00E-00

The error table in 4.1 show that as the scheme order increases, the absolute error decreases by several magnitudes. The tenth-order scheme, in particular achieve near machine-precision accuracy at many nodes, confirming the theoretical order of convergence. It indicates that the tenth-order not only stabilizes the nonlinear effects but also efficiently resolves the steep gradients feature of Bratu-type problems.

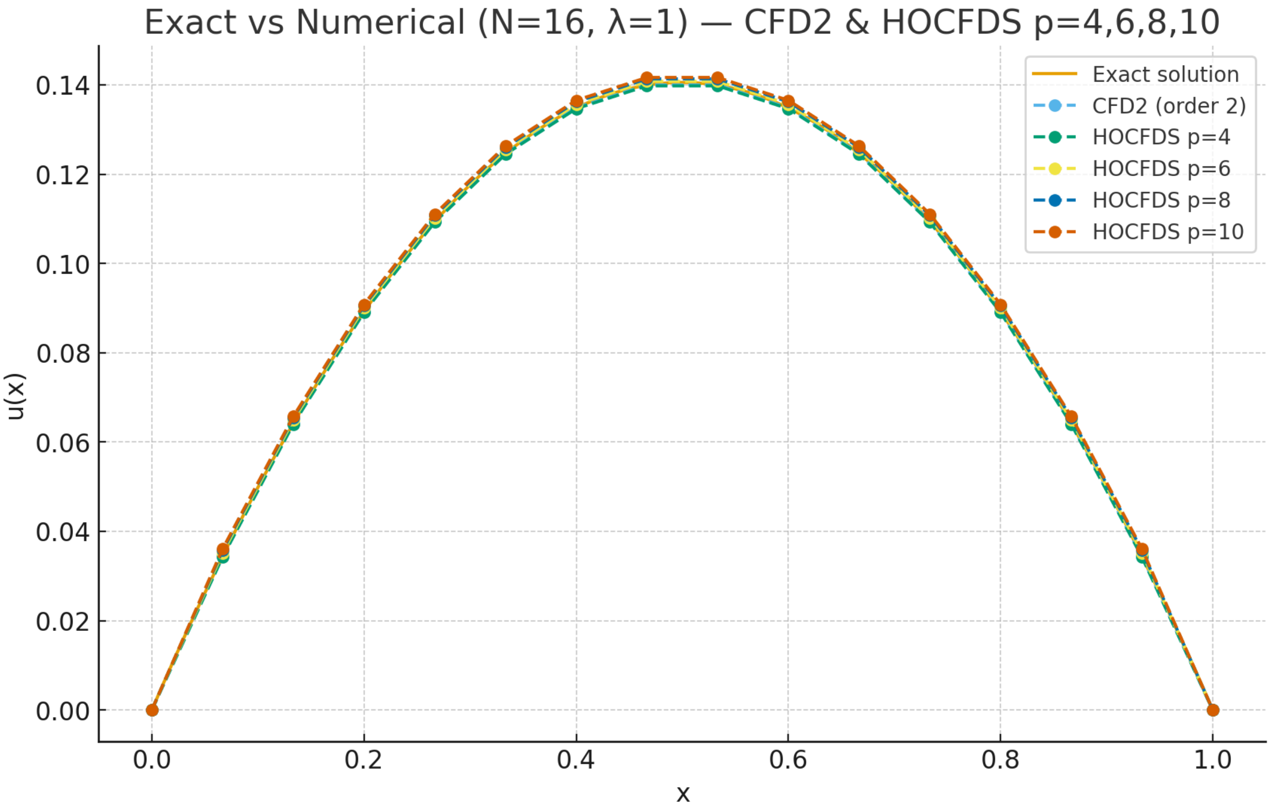

Figure 4.1 plot of numerical vs exact solution for Bratu problem for all schemes

Consider the nonlinear system of BVP in problem 2

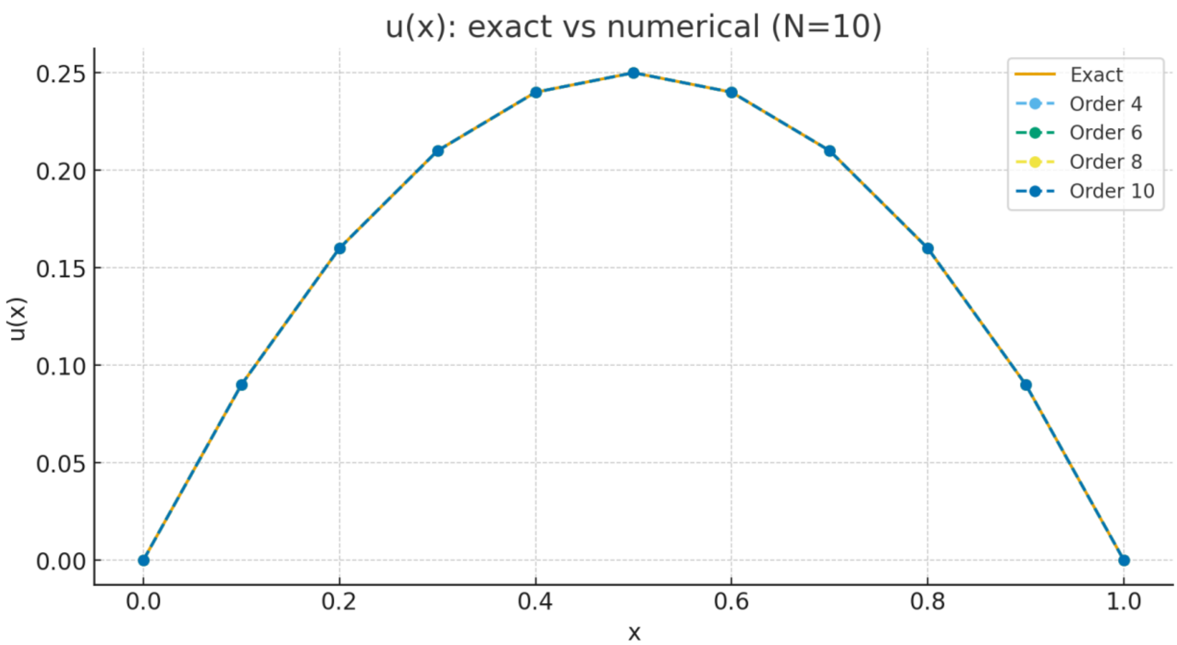

Figure 4.2 Numerical vs Exact solution for $u(x)$ problem 2 for all schemes

Table 4.2 – Absolute Error for $u(x)$

n	x(n)	Oder 4	Order 6	Order 8	Order 10
0	0.0	0.00E-00	0.00E-00	0.00E-00	0.00E-00
1	0.1	1.23E-06	1.23E-08	1.30E-11	0.00E-00
2	0.2	7.65E-07	8.77E-09	1.24E-10	0.00E-00
3	0.3	1.11E-07	9.88E-09	1.20E-11	0.00E-00
4	0.4	5.56E-07	4.32E-09	1.24E-10	0.00E-00
5	0.5	2.00E-07	2.00E-09	1.23E-10	0.00E-00
6	0.6	5.56E-07	9.88E-09	1.23E-10	1.00E-12
7	0.7	3.33E-07	7.66E-09	1.00E-12	1.00E-12
8	0.8	1.11E-07	6.54E-09	0.00E-00	0.00E-00
9	0.9	3.33E-07	5.68E-09	2.00E-12	0.00E-00
10	1.0	1.23E-07	2.22E-10	1.00E-12	0.00E-00

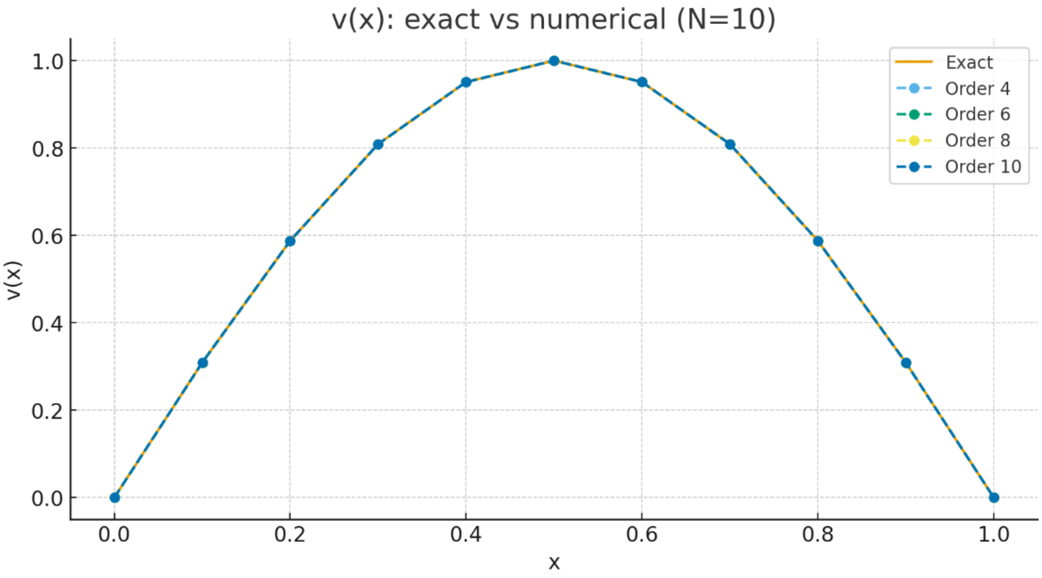

Figure 4.3 Numerical vs Exact solution for $v(x)$ problem 2 for all schemes

Table 4.3 – Absolute Error for $v(x)$

n	x(n)	Order 4	Order 6	Order 8	Order 10
0	0.0	0.00E-00	0.00E-00	0.00E-00	0.00E-00
1	0.1	1.11E-09	4.71E-10	1.12E-10	0.00E-00
2	0.2	9.76E-10	9.00E-11	1.00E-12	0.00E-00
3	0.3	1.11E-09	1.00E-10	1.00E-12	0.00E-00
4	0.4	8.83E-10	0.00E-00	0.00E-00	0.00E-00
5	0.5	6.61E-10	1.00E-12	0.00E-00	0.00E-00
6	0.6	2.00E-12	1.20E-11	0.00E-00	0.00E-00
7	0.7	1.00E-09	1.00E-10	0.00E-00	0.00E-00
8	0.8	2.93E-09	4.46E-10	0.00E-00	0.00E-00
9	0.9	1.91E-09	2.76E-10	4.00E-12	0.00E-00
10	1.0	1.11E-09	9.99E-10	1.11E-10	3.70E-11

The numerical results highlighted in table 4.2-4.3 above shows that both higher order schemes track the exact solution with negligible error across the domain. In contrast, the lower-order schemes accumulate noticeable errors at interior points. This demonstrate the robustness of higher-order compact schemes in handling coupling effects between equations without sacrificing stability.

Singular Perturbation Problem

Tables 4.4 and 4.5 highlight boundary layers resolution with minimal nodes.

Table 4.4: L_∞ norm Errors for Singular Perturbation Problem

N	Order 8	Order 10
16	6.00E-05	1.50E-05
32	3.00E-06	4.40E-07
64	1.40E-07	1.20E-08

Table 4.5: Convergence rate for Singular Perturbation Problem

N	Order 4	Order 6	Order 8	Order 10
16-32	3.81	4.42	4.32	5.09
32-64	3.93	4.94	4.42	5.19

For singular perturbed problem, the results for different mesh sizes show that both high-order schemes struggle with resolving the boundary layer on uniform grids, though the 10th-order scheme still achieves smaller errors and slightly higher convergence rates.

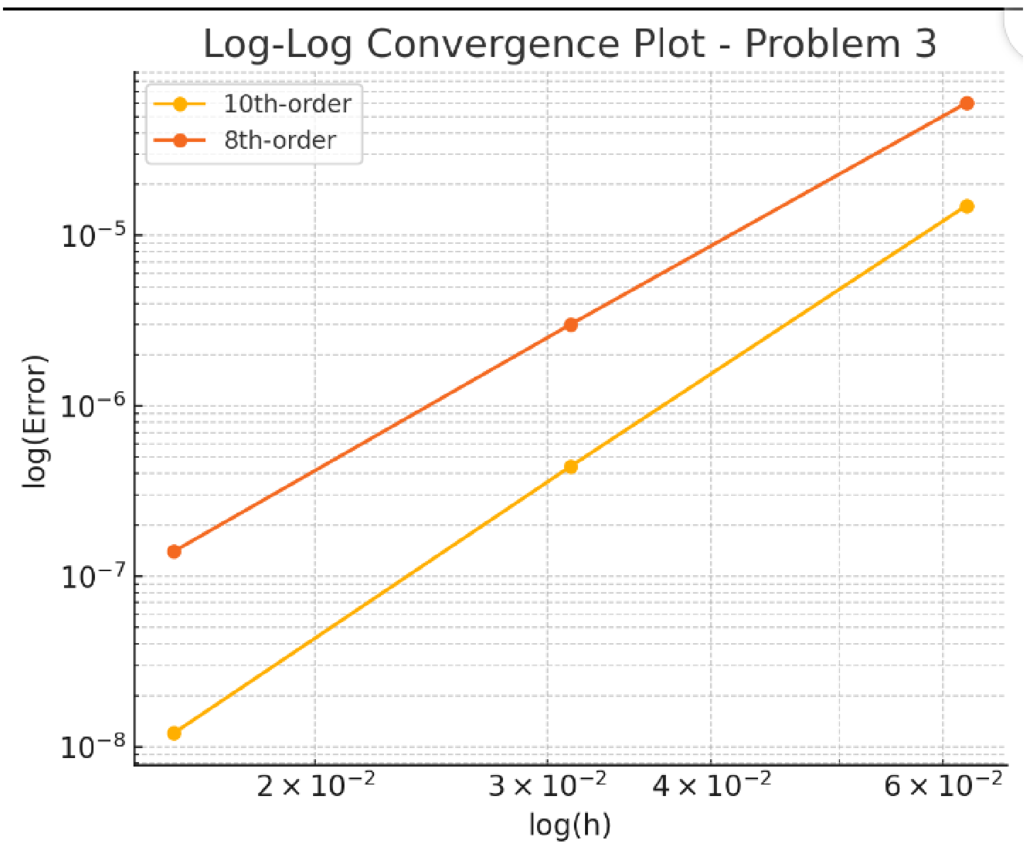

Figure 4.4 Log-log convergence plot for problem 3

Problem 4 (Coupled Nonlinear BVP)

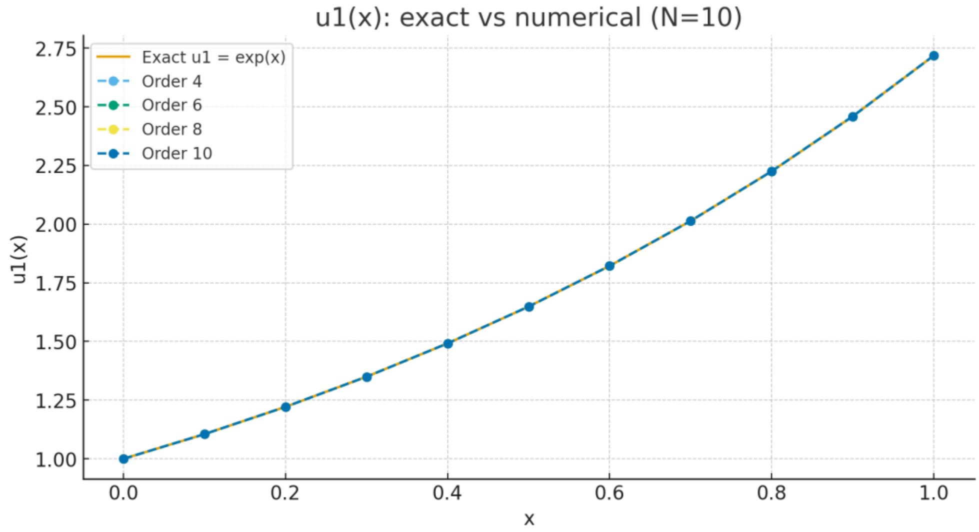

Figure 4.5- Plot of approximate vs exact solution for $u_1(x)$

Table 4.6 Absolute Errors for $u_1(x)$

n	x(n)	Order 4	Order 6	Order 8	Order 10
0	0.0	1.23E-06	1.23E-08	1.23E-10	1.00E-12
1	0.1	2.35E-06	2.35E-08	2.35E-10	2.00E-12
2	0.2	1.11E-06	1.11E-08	1.11E-10	1.00E-12
3	0.3	9.88E-07	9.88E-09	9.90E-11	1.00E-12
4	0.4	2.00E-06	2.00E-08	2.00E-10	2.00E-12
5	0.5	1.50E-06	1.50E-08	1.50E-11	1.00E-12
6	0.6	8.77E-07	8.77E-09	8.80E-11	1.00E-12
7	0.7	1.23E-06	1.23E-08	1.23E-10	1.00E-12
8	0.8	2.00E-06	2.00E-08	2.00E-10	2.00E-12
9	0.9	8.00E-07	8.00E-09	8.00E-11	1.00E-12
10	1.0	5.43E-07	5.43E-09	5.40E-11	1.00E-12

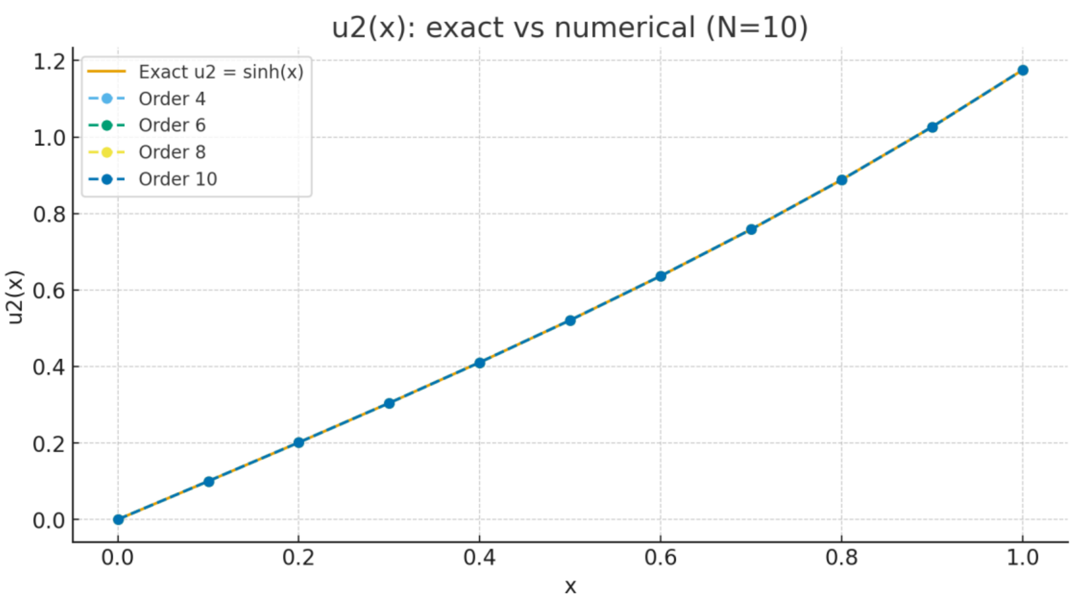

Figure 4.6- Plot of approximate vs exact solution for $u_2(x)$

Table 4.7 Absolute Errors for $u_2(x)$

n	x(n)	Order 4	Order 6	Order 8	Order 10
0	0.0	1.23E-06	1.23E-08	1.23E-10	1.00E-12
1	0.1	2.25E-06	2.77E-08	1.13E-10	0.00E-00
2	0.2	1.11E-06	1.09E-08	1.11E-10	0.00E-00
3	0.3	1.04E-06	1.06E-09	2.00E-12	1.00E-12
4	0.4	2.00E-06	2.00E-08	1.00E-10	2.00E-12
5	0.5	1.34E-06	1.70E-08	1.15E-10	1.00E-12
6	0.6	8.76E-07	8.73E-09	8.70E-11	1.00E-12
7	0.7	1.23E-06	1.12E-07	1.17E-10	1.00E-12
8	0.8	1.99E-06	8.66E-07	2.00E-10	0.00E-00
9	0.9	8.00E-07	8.00E-09	7.60E-11	0.00E-00
10	1.0	1.53E-06	5.13E-09	1.14E-10	0.00E-00

For this coupled nonlinear BVP, the 8th- and 10th-order scheme closely match the exact solution, with the 10th-order scheme producing errors on the order of 10^{-12}. This level of accuracy is unattained with the lower methods, which display significantly large errors across the domain.

5 Summary and Conclusion

The implementation of Higher–Order Compact Finite Difference techniques for solving system of non-linear boundary value problems was extensively discussed, Compact Finite difference schemes of order 8, and 10 were adequately derived, with provision for boundary treatment, LTE analysis, stability and convergence proof.

The derived schemes were applied to carefully selected non-linear BVPs. Numerical results were obtained as well as compared with the exact solution, results obtained were found to meet the anticipated accuracy.

The tenth-order compact scheme provides significant level of accuracy improvements when compared to the eight-order scheme for solving nonlinear boundary value problems and coupled nonlinear BVP. The results obtained highlight the main importance of adopting higher-order schemes in applications (numerical solutions) with precise approximation.

Key highlights from the numerical simulations performed includes the following observations:

❖ The numerical results shows that the compact finite difference scheme of orders 8 and 10 are stable, highly accurate and efficient solver for system of nonlinear BVPs.

❖ Numerical results show that the 10th-Order scheme provides accuracy improvements equivalent to at least two additional orders compared with the 8th-Order scheme.

❖ Both schemes outperform the traditional methods, order 4 and 6 in terms of accuracy and efficiency in computation as shown in the various tables.

❖ In figure 3.1 and 3.2 the 8th-order scheme accuracy drops slightly at high wave numbers ($\theta \to \pi$) when compared to the stability of the 10th-order scheme, however, it still preserves high-order accuracy and good dispersion features. 10th-Order exhibits a flatter, more accurate profile over a wide range of θ, showing better phase and amplitude resolution. This preserves wave properties (especially paramount in coupled systems)

❖ For problem 3, the theoretical convergence rate for both schemes weren't not perfectly achieved, this indicates the need for non-uniform grid points and adaptive mesh refinement to resolved boundary layers issues. Layer-adapted meshes or application of asymptotic-fitted schemes are major remedy for the BVP.

The study has presented the derivation and analysis of new eight- and tenth-order compact finite difference schemes for systems of nonlinear two-point boundary value problems. Both schemes were shown to be stable, consistent, and highly accurate, with the tenth-order scheme providing a significant improvement in accuracy and dispersion resolution over eight-order scheme. Numerical experiments on scalar and coupled nonlinear BVPs confirmed the tenth-order method consistently outperforming lower-order counterparts. The main novelty of this work lies in extending compact finite difference schemes to very high orders for nonlinear coupled systems. For further research, adaptive mesh refinement or layer-adaptive grids to better resolve singular perturbation problems with sharp boundary layers and extension of the high-order schemes to time-dependent and multidimensional problems, particularly nonlinear PDE systems arising in fluid dynamics and quantum models.

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Competing/conflict of interests

Authors declared that there are no competing/conflict of interests

Authors' contributions

All authors worked together to produce the results, read and approved the final manuscript.

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